Part II

Multiperiod Consumption, Portfolio Choice, and Asset Pricing

Chapter 5

A Multiperiod Discrete-Time Model of Consumption and Portfolio Choice

This chapter considers an expected-utility-maximizing individual's consumption and portfolio choices over many periods. In contrast to our previous single-period or static models, here the intertemporal or dynamic nature of the problem is explicitly analyzed. Solving an individual's multiperiod consumption and portfolio choice problem is of interest in that it provides a theory for an individual's optimal lifetime savings and investment strategies. Hence, it has normative value as a guide for individual financial planning. In addition, just as our single-period mean-variance portfolio selection model provided the theory of asset demands for the Capital Asset Pricing Model, a multiperiod portfolio choice model provides a theory of asset demands for a general equilibrium theory of intertemporal capital asset pricing. Combining this model of individuals' preferences over consumption and securities with a model of firm production technologies can lead to an equilibrium model of the economy that determines asset price processes.[1]

In the 1920s, Frank Ramsey (Ramsey 1928) derived optimal multiperiod consumption-savings decisions but assumed that the individual could invest in only a single asset paying a certain return. It was not until the late 1960s that Paul A. Samuelson (Samuelson 1969) and Robert C. Merton (Merton 1969) were able to solve for an individual's multiperiod consumption and portfolio choice decisions under uncertainty, that is, where both a consumption-savings choice and a portfolio allocation decision involving risky assets were assumed to occur each period.[2] Their solution technique involves stochastic dynamic programming. While this dynamic programming technique is not the only approach to solving problems of this type, it can sometimes be the most convenient and intuitive way of deriving solutions.[3]

The model we present allows an individual to make multiple consumption and portfolio decisions over a single planning horizon. This planning horizon, which can be interpreted as the individual's remaining lifetime, is composed of many decision periods, with consumption and portfolio decisions occurring once each period. The richness of this problem cannot be captured in the single-period models that we presented earlier. This is because with only one period, an investor's decision period and planning horizon coincide. Still, the results

[1] Important examples of such models were developed by John Cox, Jonathan Ingersoll, and Stephen Ross (Cox, Ingersoll, and Ross 1985a) and Robert Lucas (Lucas 1978).

[2] Jan Mossin (Mossin 1968) solved for an individual's optimal multiperiod portfolio decisions but assumed the individual had no interim consumption decisions, only a utility of terminal consumption.

[3] An alternative martingale approach to solving consumption and portfolio choice problems is given by John C. Cox and Chi-Fu Huang (Cox and Huang 1989). This approach will be presented in Chapter 12 in the context of a continuous-time consumption and portfolio choice problem.

from our single-period analysis will be useful because often we can transform multiperiod models into a series of single-period ones, as will be illustrated next.

The consumption-portfolio choice model presented in this chapter assumes that the individual's decision interval is a discrete time period. Later in this book, we change the assumption to make the interval instantaneous; that is, the individual may make consumption and portfolio choices continuously. This latter assumption often simplifies problems and can lead to sharper results. When we move from discrete time to continuous time, continuous-time stochastic processes are used to model security prices.

The next section outlines the assumptions of the individual's multiperiod consumption-portfolio problem. Perhaps the strongest assumption that we make is that utility of consumption is time separable.[4] The following section shows how this problem can be solved. It introduces an important technique for solving multiperiod decision problems under uncertainty, namely, stochastic dynamic programming. The beauty of this technique is that decisions over a multiperiod horizon can be broken up into a series of decisions over a single-period horizon. This allows us to derive the individual's optimal consumption and portfolio choices by starting at the end of the individual's planning horizon and working backwards toward the present. In the last section, we complete our analysis by deriving explicit solutions for the individual's consumption and portfolio holdings when utility is assumed to be logarithmic.

5.1 Assumptions and Notation of the Model

Consider an environment in which an individual chooses his level of consumption and the proportions of his wealth invested in n risky assets plus a risk-free

[4] Time-inseparable utility, where current utility can depend on past or expected future consumption, is discussed in Chapter 14.

asset. As was the case in our single-period models, it is assumed that the individual takes the stochastic processes followed by the prices of the different assets as given. The implicit assumption is that security markets are perfectly competitive in the sense that the (small) individual is a price-taker in security markets. An individual's trades do not impact the price (or the return) of the security. For most investors trading in liquid security markets, this is a reasonably realistic assumption. In addition, it is assumed that there are no transactions costs or taxes when buying or selling assets, so that security markets can be described as "frictionless."

An individual is assumed to make consumption and portfolio choice decisions at the start of each period during a T-period planning horizon. Each period is of unit length, with the initial date being 0 and the terminal date being T.[5]

5.1.1 Preferences

The individual is assumed to maximize an expected utility function defined over consumption levels and a terminal bequest. Denote consumption at date t as C_t, $t = 0, ..., T-1$, and the terminal bequest as W_T, where W_t indicates the individual's level of wealth at date t. A general form for a multiperiod expected utility function would be $E_0 \left[\Upsilon \left(C_0, C_1, ..., C_{T-1}, W_T \right) \right]$, where we could simply assume that Υ is increasing and concave in its arguments. However, as a starting point, we will assume that Υ has the following *time-separable,* or *additively separable*, form:

$$E_0 \left[\Upsilon \left(C_0, C_1, ..., C_{T-1}, W_T \right) \right] = E_0 \left[\sum_{t=0}^{T-1} U \left(C_t, t \right) + B \left(W_T, T \right) \right] \tag{5.1}$$

[5] The following presentation borrows liberally from Samuelson (Samuelson 1969) and Robert C. Merton's unpublished MIT course 15.433 class notes "Portfolio Theory and Capital Markets."

where U and B are assumed to be increasing, concave functions of consumption and wealth, respectively. Equation (5.1) restricts utility at date t, $U(C_t, t)$, to depend only on consumption at that date and not previous levels of consumption or expected future levels of consumption. While this is the traditional assumption in multiperiod models, in later chapters we loosen this restriction and investigate utility formulations that are not time separable.[6]

5.1.2 The Dynamics of Wealth

At date t, the value of the individual's tangible wealth held in the form of assets equals W_t. In addition, the individual is assumed to receive wage income of y_t.[7] This beginning-of-period wealth and wage income are divided between consumption and savings, and then savings is allocated between n risky assets as well as a risk-free asset. Let R_{it} be the random return on risky asset i over the period starting at date t and ending at date $t+1$. Also let R_{ft} be the return on an asset that pays a risk-free return over the period starting at date t and ending at date $t+1$. Then if the proportion of date t saving allocated to risky asset i is denoted ω_{it}, we can write the evolution of the individual's tangible wealth as

$$\begin{aligned} W_{t+1} &= (W_t + y_t - C_t) \quad \left. R_{ft} + \sum_{i=1}^{n} \omega_{it} (R_{it} - R_{ft}) \right) \qquad (5.2) \\ &= S_t R_t \end{aligned}$$

where $S_t \equiv W_t + y_t - C_t$ is the individual's savings at date t, and $R_t \equiv R_{ft} + \sum_{i=1}^{n} \omega_{it} (R_{it} - R_{ft})$ is the total return on the individual's invested wealth over

[6] Dynamic programming, the solution technique presented in this chapter, can also be applied to consumption and portfolio choice problems where an individual's utility is time inseparable.

[7] Wage income can be random. The present value of wage income, referred to as human capital, is assumed to be a nontradeable asset. The individual can rebalance how his financial wealth is allocated among risky assets but cannot trade his human capital.

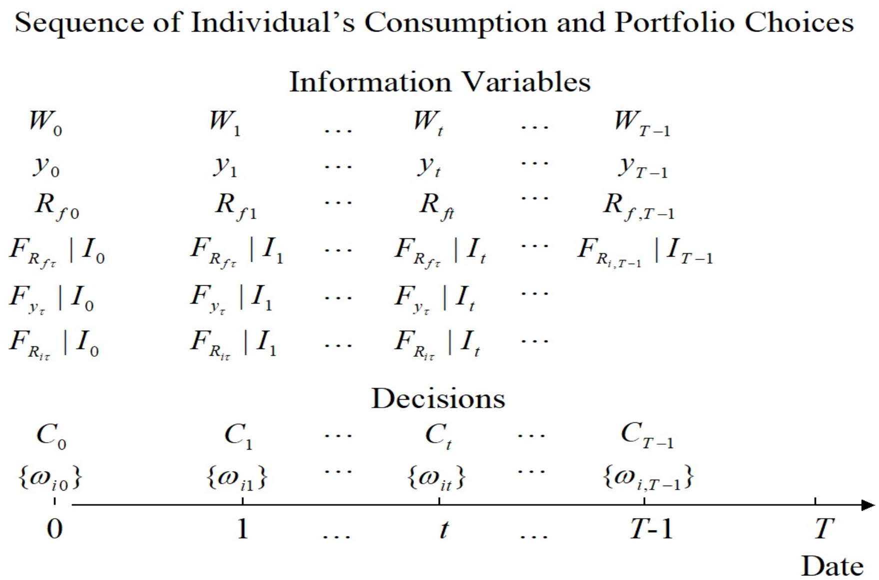

Figure 5.1: Multiperiod Decisions

the period from date t to $t+1$.

Note that we have not restricted the distribution of asset returns in any way. In particular, the return distribution of risky asset i could change over time, so that the distribution of R_{it} could differ from the distribution of $R_{i\tau}$ for $t \neq \tau$. Moreover, the one-period risk-free return could be changing, so that $R_{ft} \neq R_{f\tau}$. Asset distributions that vary from one period to the next mean that the individual faces changing investment opportunities. Hence, in a multiperiod model, the individual's current consumption and portfolio decisions may be influenced not only by the asset return distribution for the current period, but also by the possibility that asset return distributions could change in the future.

The information and decision variables available to the individual at each date are illustrated in Figure 5.1. At date t, the individual knows her wealth at the start of the period, W_t; her wage income received at date t, y_t; and the risk-free interest rate for investing or borrowing over the period from date t to date $t+1$, R_{ft}. Conditional on information at date t, denoted by I_t, she also knows the distributions of future one-period risk-free rates and wage income, $F_{R_{f\tau}}|I_t$ and $F_{y_\tau}|I_t$, respectively, for dates $\tau = t+1, ..., T-1$. Lastly, the individual also knows the date t conditional distributions of the risky-asset returns for dates $\tau = t, ..., T-1$, given by $F_{R_{i\tau}}|I_t$. Date t information, I_t, includes all realizations of wage income and risk-free rates for all dates up until and including date t. I_t also includes all realizations of risky-asset returns for all dates up until and including date $t-1$. Moreover, I_t could include any other state variables known at date t that affect the distributions of future wages, risk-free rates, and risky-asset returns. Based on this information, the individual's date t decision variables are consumption, C_t, and the portfolio weights for the n risky assets, $\{\omega_{it}\}$, for $i = 1, ..., n$.

5.2 Solving the Multiperiod Model

We begin by defining an important concept that will help us simplify the solution to this multiperiod optimization problem. Let $J(W_t, t)$ denote the derived utility-of-wealth function. It is defined as follows:

$$J(W_t, I_t, t) \equiv \max_{C_s, \{\omega_{is}\}, \forall s,i} E_t \left[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T) \right] \tag{5.3}$$

where "max" means to choose the decision variables C_s and $\{\omega_{is}\}$ for $s = t, t+1, ..., T-1$ and $i = 1, ..., n$ so as to maximize the expected value of the term in brackets. Note that J is a function of current wealth and all information up until and including date t. This information could reflect state variables describing a changing distribution of risky-asset returns and/or a changing risk-free interest rate, where these state variables are assumed to be exogenous to the individual's consumption and portfolio choices. However, by definition J is not a function of the individual's current or future decision variables, since they are assumed to be set to those values that maximize lifetime expected utility. Hence, J can be described as a "derived" utility-of-wealth function.

We will solve the individual's consumption and portfolio choice problem using backward dynamic programming. This entails considering the individual's multiperiod planning problem starting from her final set of decisions because, with one period remaining in the individual's planning horizon, the multiperiod problem has become a single-period one. We know from Chapter 4 how to solve for consumption and portfolio choices in a single-period context. Once we characterize the last period's solution for some given wealth and distribution of asset returns faced by the individual at date $T-1$, we can solve for the individual's optimal decisions for the preceding period, those decisions made at date $T-2$. This procedure is continued until we can solve for the individual's optimal decisions at the current date 0. As will be clarified next, by following this recursive solution technique, the individual's current decisions properly account for future optimal decisions that she will make in response to the evolution of uncertainty in asset returns and labor income.

5.2.1 The Final Period Solution

From the definition of J, note that[8]

$$J\left(W_T, T\right) = E_T\left[B\left(W_T, T\right)\right] = B\left(W_T, T\right) \tag{5.4}$$

Now working backwards, consider the individual's optimization problem when, at date $T-1$, she has a single period left in her planning horizon.

[8] To keep notation manageable, we suppress making information, I_t, an explicit argument of the indirect utility function. We use the shorthand notation $J\left(W_t, t\right)$ to refer to $J\left(W_t, I_t, t\right)$.

$$
\begin{aligned}
J\left(W_{T-1}, T-1\right) &= \max_{C_{T-1},\{\omega_{i,T-1}\}} E_{T-1}\left[U\left(C_{T-1}, T-1\right)+B\left(W_T, T\right)\right] \quad (5.5)\\
&= \max_{C_{T-1},\{\omega_{i,T-1}\}} U\left(C_{T-1}, T-1\right)+E_{T-1}\left[B\left(W_T, T\right)\right]
\end{aligned}
$$

To clarify how W_T depends explicitly on C_{T-1} and $\{\omega_{i,T-1}\}$, substitute equation (5.2) for $t = T-1$ into equation (5.5):

$$
J\left(W_{T-1}, T-1\right) = \max_{C_{T-1},\{\omega_{i,T-1}\}} U\left(C_{T-1}, T-1\right)+E_{T-1}\left[B\left(S_{T-1}R_{T-1}, T\right)\right] \tag{5.6}
$$

where it should be recalled that $S_{T-1} \equiv W_{T-1} + y_{T-1} - C_{T-1}$ and $R_{T-1} \equiv R_{f,T-1} + \sum_{i=1}^{n} \omega_{i,T-1}\left(R_{i,T-1} - R_{f,T-1}\right)$. Equation (5.6) is a standard single-period consumption-portfolio choice problem. To solve it, we differentiate with respect to each decision variable, C_{T-1} and $\{\omega_{i,T-1}\}$, and set the resulting expressions equal to zero:

$$
U_C\left(C_{T-1}, T-1\right) - E_{T-1}\left[B_W\left(W_T, T\right)R_{T-1}\right] = 0 \tag{5.7}
$$

$$
E_{T-1}\left[B_W\left(W_T, T\right)\left(R_{i,T-1} - R_{f,T-1}\right)\right] = 0,\ \ i = 1, ..., n \tag{5.8}
$$

where the subscripts on U and B denote partial differentiation.[9] Using the results in (5.8), we see that (5.7) can be rewritten as

$$
\begin{aligned}
U_C\left(C_{T-1}, T-1\right) &= E_{T-1}\left[B_W\left(W_T, T\right)\left(R_{f,T-1} + \sum_{i=1}^{n} \omega_{i,T-1}\left(R_{i,T-1} - R_{f,T-1}\right)\right)\right]\\
&= R_{f,T-1}E_{T-1}\left[B_W\left(W_T, T\right)\right] \quad (5.9)
\end{aligned}
$$

[9] Note that we apply the chain rule when differentiating $B\left(W_T, T\right)$ with respect to C_{T-1} since $W_T = S_{T-1}R_{T-1}$ depends on C_{T-1} through S_{T-1}.

Conditions (5.8) and (5.9) represent $n+1$ equations that determine the optimal choices of C^*_{T-1} and $\left\{\omega^*_{i,T-1}\right\}$. They are identical to the single-period model conditions (4.6) and (4.10) derived in the previous chapter but with the utility of bequest function, B, replacing the end-of-period utility function, U. If we substitute these optimal decision variables back into equation (5.6) and differentiate totally with respect to W_{T-1}, we have

$$\begin{aligned} J_W &= U_C \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \left(\frac{dW_T}{dW_{T-1}}\right)\right] \\ &= U_C \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \quad \frac{\partial W_T}{\partial W_{T-1}} + \sum_{i=1}^{n} \frac{\partial W_T}{\partial \omega^*_{i,T-1}} \frac{\partial \omega^*_{i,T-1}}{\partial W_{T-1}} \right. \\ &\quad \left. + \frac{\partial W_T}{\partial C^*_{T-1}} \frac{\partial C^*_{T-1}}{\partial W_{T-1}} \right)\Bigg] \\ &= U_C \frac{\partial C^*_{T-1}}{\partial W_{T-1}} + E_{T-1}\left[B_{W_T} \cdot \quad \sum_{i=1}^{n} \left[R_{i,T-1} - R_{f,T-1}\right] S_{T-1} \frac{\partial \omega^*_{i,T-1}}{\partial W_{T-1}} \right. \\ &\quad \left. + R_{T-1}\left(1 - \frac{\partial C^*_{T-1}}{\partial W_{T-1}}\right)\right)\Bigg] \end{aligned} \tag{5.10}$$

Using the first-order condition (5.8), $E_{T-1}\left[B_{W_T} \cdot (R_{i,T-1} - R_{f,T-1})\right] = 0$, as well as (5.9), $U_C = R_{f,T-1} E_{T-1}\left[B_{W_T}\right]$, we see that (5.10) simplifies to $J_W = R_{f,T-1} E_{T-1}\left[B_{W_T}\right]$. Using (5.9) once again, this can be rewritten as

$$J_W\left(W_{T-1}, T-1\right) = U_C\left(C^*_{T-1}, T-1\right) \tag{5.11}$$

which is known as the "envelope condition." It says that the individual's optimal policy equates her marginal utility of current consumption, U_C, to her marginal utility of wealth (future consumption).

5.2.2 Deriving the Bellman Equation

Having solved the individual's problem with one period to go in her planning horizon, we next consider her optimal consumption and portfolio choices with two periods remaining, at date $T-2$. The individual's objective at this date is

$$\begin{aligned} J\left(W_{T-2}, T-2\right) &= \max U\left(C_{T-2}, T-2\right) + E_{T-2}\left[U\left(C_{T-1}, T-1\right)\right. \\ &\quad \left. + B\left(W_T, T\right)\right] \end{aligned} \tag{5.12}$$

The individual must maximize expression (5.12) by choosing C_{T-2} as well as $\{\omega_{i,T-2}\}$. However, note that she wishes to maximize an expression that is an expectation over utilities $U\left(C_{T-1}, T-1\right) + B\left(W_T, T\right)$ that depend on future decisions, namely, C_{T-1} and $\{\omega_{i,T-1}\}$. What should the individual assume these future values of C_{T-1} and $\{\omega_{i,T-1}\}$ to be? The answer comes from the *Principle of Optimality.* It states:

> An optimal set of decisions has the property that given an initial decision, the remaining decisions must be optimal with respect to the outcome that results from the initial decision.

The "max" in (5.12) is over all remaining decisions, but the Principle of Optimality says that whatever decision is made in period $T-2$, given the outcome, the remaining decisions (for period $T-1$) must be optimal (maximal). In other words,

$$\max_{\{(T-2),(T-1)\}} (Y) = \max_{\{T-2\}} \left[\max_{\{T-1, \mid \text{ outcome from } (T-2)\}} (Y) \right] \tag{5.13}$$

This principle allows us to rewrite (5.12) as

$$\begin{aligned} J\left(W_{T-2}, T-2\right) &= \max_{C_{T-2}, \{\omega_{i,T-2}\}} \left\{ U\left(C_{T-2}, T-2\right) + \right. \\ &\left. E_{T-2}\left[\max_{C_{T-1}, \{\omega_{i,T-1}\}} E_{T-1}\left[U\left(C_{T-1}, T-1\right) + B\left(W_T, T\right) \right] \right] \right\} \end{aligned} \tag{5.14}$$

Then, using the definition of $J\left(W_{T-1}, T-1\right)$ from (5.5), equation (5.14) can be rewritten as

$$J\left(W_{T-2}, T-2\right) = \max_{C_{T-2}, \{\omega_{i,T-2}\}} U\left(C_{T-2}, T-2\right) + E_{T-2}\left[J\left(W_{T-1}, T-1\right) \right] \tag{5.15}$$

The recursive condition (5.15) is known as the (Richard) Bellman equation (Bellman 1957). It characterizes the individual's objective at date $T-2$. What is important about this characterization is that if we compare it to equation (5.5), the individual's objective at date $T-1$, the two problems are quite similar. The only difference is that in (5.15) we replace the known function of wealth next period, B, with another (known in principle) function of wealth next period, J. But the solution to (5.15) will be of the same form as that for (5.5).[10]

5.2.3 The General Solution

Thus, the optimality conditions for (5.15) are

[10] Using the envelope condition, it can be shown that the concavity of U and B ensures that $J\left(W, t\right)$ is a concave and continuously differentiable function of W. Hence, an interior solution to the second-to-last period problem exists.

$$
\begin{aligned}
U_C\left(C_{T-2}^*, T-2\right) &= E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) R_{T-2}\right] \\
&= R_{f,T-2} E_{T-2}\left[J_W\left(W_{T-1}, T-1\right)\right] \\
&= J_W\left(W_{T-2}, T-2\right)
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
E_{T-2}\left[R_{i,T-2} J_W\left(W_{T-1}, T-1\right)\right] &= R_{f,T-2} E_{T-2}\left[J_W\left(W_{T-1}, T-1\right)\right], \\
i &= 1, ..., n
\end{aligned}
\tag{5.17}
$$

Based on the preceding pattern, inductive reasoning implies that for any $t = 0, 1, ..., T-1$, we have the Bellman equation:

$$
J\left(W_t, t\right) = \max_{C_t, \{\omega_{i,t}\}} U\left(C_t, t\right) + E_t\left[J\left(W_{t+1}, t+1\right)\right] \tag{5.18}
$$

and, therefore, the date t optimality conditions are

$$
\begin{aligned}
U_C\left(C_t^*, t\right) &= E_t\left[J_W\left(W_{t+1}, t+1\right) R_t\right] \\
&= R_{f,t} E_t\left[J_W\left(W_{t+1}, t+1\right)\right] \\
&= J_W\left(W_t, t\right)
\end{aligned}
\tag{5.19}
$$

$$
E_t\left[R_{i,t} J_W\left(W_{t+1}, t+1\right)\right] = R_{f,t} E_t\left[J_W\left(W_{t+1}, t+1\right)\right], \; i = 1, ..., n \tag{5.20}
$$

The insights of the multiperiod model conditions (5.19) and (5.20) are similar to those of a single-period model from Chapter 4. The individual chooses today's consumption such that the marginal utility of current consumption equals the derived marginal utility of wealth (the marginal utility of future consumption). Furthermore, the portfolio weights should be adjusted to equate all assets' expected marginal utility-weighted asset returns. However, solving for the individual's actual consumption and portfolio weights at each date, C_t^* and $\{\omega_{i,t}\}$, $t = 0, ..., T-1$, is more complex than for a single-period model. The conditions' dependence on the derived utility-of-wealth function implies that they depend on future contingent investment opportunities (the distributions of future asset returns ($R_{i,t+j}, R_{f,t+j}$, $j \geq 1$), future income flows, y_{t+j}, and possibly, states of the world that might affect future utilities ($U(\cdot, t+j)$).

Solving this system involves starting from the end of the planning horizon and dynamically programing backwards toward the present. Thus, for the last period, T, we know that $J(W_T, T) = B(W_T, T)$. As we did previously, we substitute $B(W_T, T)$ for $J(W_T, T)$ in conditions (5.18) to (5.20) for date $T-1$ and solve for $J(W_{T-1}, T-1)$. This is then substituted into conditions (5.18) to (5.20) for date $T-2$ and one then solves for $J(W_{T-2}, T-2)$. If we proceed in this recursive manner, we eventually obtain $J(W_0, 0)$ and the solution is complete. These steps are summarized in the following table.

Step	Action
1	Construct $J(W_T, T)$.
2	Solve for C_{T-1}^* and $\{\omega_{i,T-1}\}$, $i = 1, ..., n$.
3	Substitute the decisions in step 2 to construct $J(W_{T-1}, T-1)$.
4	Solve for C_{T-2}^* and $\{\omega_{i,T-2}\}$, $i = 1, ..., n$.
5	Substitute the decisions in step 4 to construct $J(W_{T-2}, T-2)$.
6	Repeat steps 4 and 5 for date $T-3$.
7	Repeat step 6 for all prior dates until date 0 is reached.

By following this recursive procedure, we find that the optimal policy will be of the form[11]

$$C_t^* = g\left[W_t, y_t, I_t, t\right] \tag{5.21}$$

$$\omega_{it}^* = h\left[W_t, y_t, I_t, t\right] \tag{5.22}$$

Deriving analytical expressions for the functions g and h is not always possible, in which case numerical solutions satisfying the first-order conditions at each date can be computed. However, for particular assumptions regarding the form of utility, wage income, and the distribution of asset returns, such explicit solutions may be possible. The next section considers an example where this is the case.

5.3 Example Using Log Utility

To illustrate how solutions of the form (5.21) and (5.22) can be obtained, consider the following example where the individual has log utility and no wage income. Assume that $U(C_t, t) \equiv \delta^t \ln[C_t]$, $B(W_T, T) \equiv \delta^T \ln[W_T]$, and $y_t \equiv 0$ $\forall$ t, where $\delta = \frac{1}{1+\rho}$ and ρ is the individual's subjective rate of time preference. Now at date $T-1$, using condition (5.7), we have

[11] When asset returns are serially correlated, that is, the date t distribution of asset returns depends on realized asset returns from periods prior to date t, the decision rules in (5.21) and (5.22) may depend on this prior, conditioning information. They will also depend on any other state variables known at time t and included in the date t information set I_t. This, however, does not affect the general solution technique. These prior asset returns are exogenous state variables that influence only the conditional expectations in the optimality conditions (5.19) and (5.20).

$$
\begin{aligned}
U_C\left(C_{T-1}, T-1\right) &= E_{T-1}\left[B_W\left(W_T, T\right) R_{T-1}\right] \qquad (5.23)\\
\delta^{T-1} \frac{1}{C_{T-1}} &= E_{T-1}\left[\delta^T \frac{R_{T-1}}{W_T}\right] = E_{T-1}\left[\delta^T \frac{R_{T-1}}{S_{T-1} R_{T-1}}\right] \\
&= \frac{\delta^T}{S_{T-1}} = \frac{\delta^T}{W_{T-1} - C_{T-1}}
\end{aligned}
$$

or

$$
C_{T-1}^{*} = \frac{1}{1+\delta} W_{T-1} \qquad (5.24)
$$

It is noteworthy that consumption for this log utility investor is a fixed proportion of wealth and is independent of investment opportunities, that is, independent of the distribution of asset returns. This is reminiscent of the result derived in Chapters 1 and 4: the income and substitution effects from a change in investment returns exactly offset each other for the log utility individual.

Turning to the first-order conditions with respect to the portfolio weights, conditions (5.8) imply

$$
\begin{aligned}
E_{T-1}\left[B_{W_T} R_{i,T-1}\right] &= R_{f,T-1} E_{T-1}\left[B_{W_T}\right],\ i = 1, ..., n \\
\delta^T E_{T-1}\left[\frac{R_{i,T-1}}{S_{T-1} R_{T-1}}\right] &= \delta^T R_{f,T-1} E_{T-1}\left[\frac{1}{S_{T-1} R_{T-1}}\right] \\
E_{T-1}\left[\frac{R_{i,T-1}}{R_{T-1}}\right] &= R_{f,T-1} E_{T-1}\left[\frac{1}{R_{T-1}}\right] \qquad (5.25)
\end{aligned}
$$

Furthermore, for the case of log utility we see that equation (5.25) equals unity, since from (5.9) we have

$$
\begin{aligned}
U_C\left(C_{T-1}, T-1\right) &= R_{f,T-1} E_{T-1}\left[B_W\left(W_T, T\right)\right] \\
\frac{\delta^{T-1}}{C_{T-1}^*} &= R_{f,T-1} E_{T-1}\left[\delta^T \frac{1}{S_{T-1} R_{T-1}}\right] \\
1 &= \frac{\delta C_{T-1}^* R_{f,T-1}}{W_{T-1} - C_{T-1}^*} E_{T-1}\left[\frac{1}{R_{T-1}}\right] \\
1 &= R_{f,T-1} E_{T-1}\left[\frac{1}{R_{T-1}}\right]
\end{aligned}
\tag{5.26}
$$

where we have substituted equation (5.24) in going from the third to the fourth line of (5.26). While we would need to make specific assumptions regarding the distribution of asset returns in order to derive the portfolio weights $\{\omega_{i,T-1}^*\}$ satisfying (5.25), note that the conditions in (5.25) are rather special in that they do not depend on W_{T-1}, C_{T-1}, or δ, but only on the particular distribution of asset returns that one assumes. The implication is that a log utility investor chooses assets in the same relative proportions, independent of his initial wealth. This, of course, is a consequence of log utility being a special case of constant relative-risk-aversion utility.[12]

The next step is to solve for $J\left(W_{T-1}, T-1\right)$ by substituting in the date $T-1$ optimal consumption and portfolio rules into the individual's objective function. Denoting $R_t^* \equiv R_{f,t} + \sum_{i=1}^n \omega_{it}^* \left(R_{it} - R_{ft}\right)$ as the individual's total portfolio return when assets are held in the optimal proportions, we have

[12] Recall from section 1.3 that a one-period investor with constant relative risk aversion places constant proportions of wealth in a risk-free and a single risky asset.

$$\begin{aligned} J\left(W_{T-1}, T-1\right) &= \delta^{T-1} \ln\left[C_{T-1}^{*}\right] + \delta^{T} E_{T-1}\left[\ln\left[R_{T-1}^{*}\left(W_{T-1} - C_{T-1}^{*}\right)\right]\right] \\ &= \delta^{T-1}\left(-\ln\left[1+\delta\right] + \ln\left[W_{T-1}\right]\right) + \\ &\quad \delta^{T}\left(E_{T-1}\left[\ln\left[R_{T-1}^{*}\right]\right] + \ln\left[\frac{\delta}{1+\delta}\right] + \ln\left[W_{T-1}\right]\right) \\ &= \delta^{T-1}\left[\left(1+\delta\right)\ln\left[W_{T-1}\right] + H_{T-1}\right] \end{aligned} \tag{5.27}$$

where $H_{T-1} \equiv -\ln\left[1+\delta\right] + \delta \ln\left[\frac{\delta}{1+\delta}\right] + \delta E_{T-1}\left[\ln\left[R_{T-1}^{*}\right]\right]$. Notably, from equation (5.25) we saw that $\omega_{i,T-1}^{*}$ did not depend on W_{T-1}, and therefore R_{T-1}^{*} and H_{T-1} do not depend on W_{T-1}.

Next, let's move back one more period and consider the individual's optimal consumption and portfolio decisions at time $T-2$. From equation (5.15) we have

$$\begin{aligned} J\left(W_{T-2}, T-2\right) &= \max_{C_{T-2}, \{\omega_{i,T-2}\}} U\left(C_{T-2}, T-2\right) + E_{T-2}\left[J\left(W_{T-1}, T-1\right)\right] \\ &= \max_{C_{T-2}, \{\omega_{i,T-2}\}} \delta^{T-2} \ln\left[C_{T-2}\right] \\ &\quad + \delta^{T-1} E_{T-2}\left[\left(1+\delta\right)\ln\left[W_{T-1}\right] + H_{T-1}\right] \end{aligned} \tag{5.28}$$

Thus, using (5.16), the optimality condition for consumption is

$$\begin{aligned} U_C\left(C_{T-2}^{*}, T-2\right) &= E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) R_{T-2}\right] \\ \frac{\delta^{T-2}}{C_{T-2}} &= \left(1+\delta\right)\delta^{T-1} E_{T-2}\left[\frac{R_{T-2}}{S_{T-2} R_{T-2}}\right] \\ &= \frac{\left(1+\delta\right)\delta^{T-1}}{W_{T-2} - C_{T-2}} \end{aligned} \tag{5.29}$$

or

$$C_{T-2}^{*}=\frac{1}{1+\delta+\delta^{2}}W_{T-2} \tag{5.30}$$

Using (5.17), we then see that the optimality conditions for $\{\omega_{i,T-2}^{*}\}$ turn out to be of the same form as at $T-1$:

$$E_{T-2}\left[\frac{R_{i,T-2}}{R_{T-2}^{*}}\right]=R_{f,T-2}E_{T-2}\left[\frac{1}{R_{T-2}^{*}}\right],\ i=1,...,n \tag{5.31}$$

and, as in the case of $T-1$, equation (5.31) equals unity, since

$$\begin{aligned}
U_C\left(C_{T-2},T-2\right) &= R_{f,T-2}E_{T-2}\left[J_W\left(W_{T-1},T-1\right)\right] \\
\frac{\delta^{T-2}}{C_{T-2}^{*}} &= R_{f,T-2}\delta^{T-1}E_{T-2}\left[\frac{1+\delta}{S_{T-2}R_{T-2}}\right] \\
1 &= \frac{\delta\left(1+\delta\right)C_{T-2}^{*}R_{f,T-2}}{W_{T-2}-C_{T-2}^{*}}E_{T-2}\left[\frac{1}{R_{T-2}}\right] \\
1 &= R_{f,T-2}E_{T-2}\left[\frac{1}{R_{T-2}}\right] \qquad (5.32)
\end{aligned}$$

Recognizing the above pattern, we see that the optimal consumption and portfolio rules for any prior date, t, are

$$C_{t}^{*}=\frac{1}{1+\delta+...+\delta^{T-t}}W_{t}=\frac{1-\delta}{1-\delta^{T-t+1}}W_{t} \tag{5.33}$$

$$E_{t}\left[\frac{R_{i,t}}{R_{t}^{*}}\right]=R_{ft}E_{t}\left[\frac{1}{R_{t}^{*}}\right]=1,\ i=1,...,n \tag{5.34}$$

Hence, we find that the consumption and portfolio rules are separable for a log utility individual. Equation (5.33) shows that the consumption-savings decision does not depend on the distribution of asset returns. Moreover, equation (5.34) indicates that the optimal portfolio proportions depend only on the distribution of one-period returns and not on the distribution of asset returns beyond the current period. This is described as *myopic behavior* because investment allocation decisions made by the multiperiod log investor are identical to those of a one-period log investor. Hence, the log utility individual's current period decisions are independent of the possibility of changing investment opportunities in future periods. It should be emphasized that these independence results are highly specific to the log utility assumption and do not occur with other utility functions. In general, it will be optimal for the individual to choose today's portfolio in a way that hedges against possible changes in tomorrow's investment opportunities. Such hedging demands for assets will become transparent when in Chapter 12 we consider the individual's consumption and portfolio choice problem in a continuous-time setting.

The consumption rule (5.33) shows that consumption is positive whenever wealth is. Since utility of consumption is undefined for logarithmic (or any other constant relative-risk-aversion) utility when consumption is nonpositive, what ensures that wealth is always positive? The individual's optimal portfolio choices will reflect this concern. While this example has not specified a specific distribution for asset returns, portfolio decisions in a discrete-time model can be quite sensitive to the requirement that wealth exceed zero. For example, suppose that the distribution of a risky asset's return had no lower bound, as would be the case if the distribution were normal. With logarithmic utility, the optimality conditions (5.34) imply that the individual avoids holding any normally distributed risky asset, since there is positive probability that a large negative return would make wealth negative as well.[13] In Chapter 12, we revisit the individual's intertemporal consumption and portfolio choices in a continuous-time environment. There we will see that the individual's ability to continuously reallocate her portfolio can lead to fundamental differences in asset demands. Individuals can maintain positive wealth even though they

[13] Note that this would not be the case for a risky asset having a return distribution that is bounded at zero, such as the lognormal distribution.

hold assets having returns that are instantaneously normally distributed. The intuition behind this difference in the discrete- versus continuous-time results is that the probability of wealth becoming negative decreases when the time interval between portfolio revisions decreases.

5.4 Summary

An individual's optimal strategy for making lifetime consumption-savings and portfolio allocation decisions is a topic having practical importance to financial planners. This chapter's analysis represents a first step in formulating and deriving a lifetime financial plan. We showed that an individual could approach this problem by a backward dynamic programming technique that first considered how decisions would be made when he reached the end of his planning horizon. For prior periods, consumption and portfolio decisions were derived using the recursive Bellman equation which is based on the concept of a derived utility of wealth function. The multiperiod planning problem was transformed into a series of easier-to-solve one-period problems. While the consumption-portfolio choice problem in this chapter assumed that lifetime utility was time separable, in future chapters we show that the Bellman equation solution technique often can apply to cases of time-inseparable lifetime utility.

Our general solution technique was illustrated for the special case of an individual having logarithmic utility and no wage income. It turned out that this individual's optimal consumption decision was to consume a proportion of wealth each period, where the proportion was a function of the remaining periods in the individual's planning horizon but not of the current or future distributions of asset returns. In other words, future investment opportunities did not affect the individual's current consumption-savings decision. Optimal portfolio allocations were also relatively simple because they depended only on the current period's distribution of asset returns.

Deriving an individual's intertemporal consumption and portfolio decisions has value beyond the application to financial planning. By summing all individuals' demands for consumption and assets, a measure of aggregate consumption and asset demands can be derived. When coupled with a theory of production technologies and asset supplies, these aggregate demands can provide the foundation for a general equilibrium theory of asset pricing. We turn to this topic in the next chapter.

5.5 Exercises

1. Consider the following consumption and portfolio choice problem. Assume that $U(C_t, t) = \delta^t \left[aC_t - bC_t{}^2\right]$, $B(W_T, T) = 0$, and $y_t \neq 0$, where $\delta = \frac{1}{1+\rho}$ and $\rho \geq 0$ is the individual's subjective rate of time preference. Further, assume that $n = 0$ so that there are no risky assets but there is a single-period riskless asset yielding a return of $R_{ft} = 1/\delta$ that is constant each period (equivalently, the risk-free interest rate $r_f = \rho$). Note that in this problem labor income is stochastic and there is only one (riskless) asset for the individual consumer-investor to hold. Hence, the individual has no portfolio choice decision but must decide only what to consume each period. In solving this problem, assume that the individual's optimal level of consumption remains below the "bliss point" of the quadratic utility function, that is, $C_t^* < \frac{1}{2}a/b, \forall t$.

a. Write down the individual's wealth accumulation equation from period t to period $t+1$.

b. Solve for the individual's optimal level of consumption at date $T-1$ and evaluate $J(W_{T-1}, T-1)$. Hint: this is trivial.

c. Continue to solve the individual's problem at date $T-2$, $T-3$, and so on - and notice the pattern that emerges. From these results, solve for the individual's optimal level of consumption for any arbitrary date, $T-t$, in terms of the individual's expected future levels of income.

2. Consider the consumption and portfolio choice problem with power utility $U(C_t, t) \equiv \delta^t C_t^\gamma / \gamma$ and a power bequest function $B(W_T, T) \equiv \delta^T W_T^\gamma / \gamma$. Assume there is no wage income ($y_t \equiv 0 \ \forall \ t$) and a constant risk-free return equal to $R_{ft} = R_f$. Also, assume that $n = 1$ and the return of the single risky asset, R_{rt}, is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

a. Derive the first-order conditions for the optimal consumption level and portfolio weight at date $T-1$, C_{T-1}^* and ω_{T-1}^*, and give an explicit expression for C_{T-1}^*.

b. Solve for the form of $J(W_{T-1}, T-1)$.

c. Derive the first-order conditions for the optimal consumption level and portfolio weight at date $T-2$, C_{T-2}^* and ω_{T-2}^*, and give an explicit expression for C_{T-2}^*.

d. Solve for the form of $J(W_{T-2}, T-2)$. Based on the pattern for $T-1$ and $T-2$, provide expressions for the optimal consumption and portfolio weight at any date $T-t$, $t = 1, 2, 3, ...$.

3. Consider the multiperiod consumption and portfolio choice problem

$$\max_{C_s, \omega_s \forall s} E_t \left[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T) \right]$$

Assume negative exponential utility $U(C_s, s) \equiv -\delta^s e^{-bC_s}$ and a bequest function $B(W_T, T) \equiv -\delta^T e^{-bW_T}$ where $\delta = e^{-\rho}$and $\rho > 0$ is the (continuously compounded) rate of time preference. Assume there is no wage income ($y_s \equiv 0 \ \forall \ s$) and a constant risk-free return equal to $R_{fs} = R_f$. Also, assume that $n = 1$ and the return of the single risky asset, R_{rs}, has an identical and independent normal distribution of $N\left(\overline{R}, \sigma^2\right)$ each period. Denote the proportion of wealth invested in the risky asset at date s as ω_s.

a. Derive the optimal portfolio weight at date $T-1$, ω^*_{T-1}. Hint: it might be easiest to evaluate expectations in the objective function prior to taking the first-order condition.

b. Solve for the optimal level of consumption at date $T-1$, C^*_{T-1}. C^*_{T-1} will be a function of W_{T-1}, b, ρ, R_f, $\overline{R}$, and σ^2.

c. Solve for the indirect utility function of wealth at date $T-1$, $J(W_{T-1}, T-1)$.

d. Derive the optimal portfolio weight at date $T-2$, ω^*_{T-2}.

e. Solve for the optimal level of consumption at date $T-2$, C^*_{T-2}.

4. An individual faces the following consumption and portfolio choice problem:

$$\max_{C_t, \omega_t \forall t} E_0 \left[\sum_{t=0}^{T-1} \delta^t \ln\left[C_t\right] + \delta^T \ln\left[W_T\right] \right]$$

where each period the individual can choose between a risk-free asset paying a time-varying return of R_{ft} over the period from t to $t+1$ and a single risky asset. The individual receives no wage income. The risky asset's return over the period from t to $t+1$ is given by

$$R_{rt} = \begin{cases} (1+u_t)\,R_{ft} & \text{with probability } \frac{1}{2} \\ (1+d_t)\,R_{ft} & \text{with probability } \frac{1}{2} \end{cases}$$

where $u_t > 0$ and $-1 < d_t < 0$. Let ω_t be the individual's proportion of wealth invested in the risky asset at date t. Solve for the individual's optimal portfolio weight ω_t^* for $t = 0, ..., T-1$.

Chapter 6

Multiperiod Market Equilibrium

The previous chapter showed how stochastic dynamic programming can be used to solve for an individual's optimal multiperiod consumption and portfolio decisions. In general, deriving an individual's decision rules for particular forms of utility and distributions of asset returns can be complex. However, even though simple solutions for individuals' decision rules may not exist, a number of insights regarding equilibrium asset pricing relationships often can be derived for an economy populated by such optimizing individuals. This is the topic of the first section of this chapter. Similar to what was shown in the context of Chapter 4's single-period consumption-portfolio choice model, here we find that an individual's first-order conditions from the multiperiod problem can be reinterpreted as equilibrium conditions for asset prices. This leads to empirically testable implications even when analytical expressions for the individuals' lifetime consumption and portfolio decisions cannot be derived. As we shall see, these equilibrium implications generalize those that we derived earlier for a single-period environment.

In the second section, we consider an important and popular equilibrium asset pricing model derived by Nobel laureate Robert E. Lucas (Lucas 1978). It is an endowment economy model of infinitely lived, representative individuals. The assumptions of the model, which determine individuals' consumption process, are particularly convenient for deriving the equilibrium price of the market portfolio of all assets. As will be shown, the model's infinite horizon gives rise to the possibility of speculative bubbles in asset prices. The last section of the chapter examines the nature of rational bubbles and considers what conditions could give rise to these nonfundamental price dynamics.

6.1 Asset Pricing in the Multiperiod Model

Recall that the previous chapter's Samuelson-Merton model of multiperiod consumption and portfolio choices assumed that an individual's objective was

$$\max_{C_s,\{\omega_{is}\},\forall s,i} E_t\left[\sum_{s=t}^{T-1} U\left(C_s,s\right)+B\left(W_T,T\right)\right] \tag{6.1}$$

and that this problem of maximizing time-separable, multiperiod utility could be transformed into a series of one-period problems where the individual solved the Bellman equation:

$$J\left(W_t,t\right)=\max_{C_t,\{\omega_{i,t}\}} U\left(C_t,t\right)+E_t\left[J\left(W_{t+1},t+1\right)\right] \tag{6.2}$$

This led to the first-order conditions

$$\begin{aligned} U_C\left(C_t^*,t\right) &= R_{f,t}E_t\left[J_W\left(W_{t+1},t+1\right)\right] \\ &= J_W\left(W_t,t\right) \end{aligned} \tag{6.3}$$

$$E_t\left[R_{it}J_W\left(W_{t+1},t+1\right)\right]=R_{f,t}E_t\left[J_W\left(W_{t+1},t+1\right)\right],\ i=1,...,n \tag{6.4}$$

By making specific assumptions regarding the form of the utility function, the nature of wage income, and the distributions of asset returns at each date, explicit formulas for C_t^* and ω_{it}^* may be derived using backward dynamic programming. The previous chapter provided an example of such a derivation for the case of an individual with log utility and no wage income. However, under more general assumptions, the multiperiod model may have equilibrium implications even when analytical expressions for consumption and portfolio choices are not possible. This is the topic that we now consider.

6.1.1 The Multi-Period Pricing Kernel

Let us illustrate how equilibrium asset pricing implications can be derived from the individual's envelope condition (6.3), $U_C\left(C_t^*,t\right)=J_W\left(W_t,t\right)$. This condition conveys that under an optimal policy, the marginal value of financial wealth equals the marginal utility of consumption. Substituting the envelope condition evaluated at date $t+1$ into the right-hand side of the first line of (6.3), we have

$$\begin{aligned}U_C\left(C_t^*,t\right) &= R_{f,t}E_t\left[J_W\left(W_{t+1},t+1\right)\right]\\ &= R_{f,t}E_t\left[U_C\left(C_{t+1}^*,t+1\right)\right]\end{aligned}\tag{6.5}$$

Furthermore, substituting (6.4) into (6.3) and, again, using the envelope condition at date $t+1$ allows us to write

$$\begin{aligned} U_C\left(C_t^*, t\right) &= E_t\left[R_{it} J_W\left(W_{t+1}, t+1\right)\right] \\ &= E_t\left[R_{it} U_C\left(C_{t+1}^*, t+1\right)\right] \end{aligned} \tag{6.6}$$

or

$$\begin{aligned} 1 &= E_t\left[m_{t,t+1} R_{it}\right] \\ &= R_{f,t} E_t\left[m_{t,t+1}\right] \end{aligned} \tag{6.7}$$

where $m_{t,t+1} \equiv U_C\left(C_{t+1}^*, t+1\right) / U_C\left(C_t^*, t\right)$ is the stochastic discount factor, or pricing kernel, between dates t and $t+1$. Equation (6.7) indicates that our previous asset pricing results derived from a single-period consumption-portfolio choice problem, such as equation (4.18), hold on a period-by-period basis even when we allow the consumption-portfolio choice problem to be a more complex multiperiod one. As before, we can interpret (6.6) and (6.7) as showing that the marginal rate of substitution between consumption at any two dates, such as t and $t+1$, equals the marginal rate of transformation. Consumption at date t can be "transformed" into consumption at date $t+1$ by investing in the riskless asset having return $R_{f,t}$ or by investing in a risky asset having the random return R_{it}.

A similar relationship can be derived for asset returns for any holding period, not just one of unit length. Note that if equation (6.6) for risky asset j is updated one period, $U_C\left(C_{t+1}^*, t+1\right) = E_{t+1}\left[R_{j,t+1} U_C\left(C_{t+2}^*, t+2\right)\right]$, and this is then substituted into the right-hand side of the original (6.6), one obtains

$$\begin{aligned} U_C\left(C_t^*, t\right) &= E_t\left[R_{it} E_{t+1}\left[R_{j,t+1} U_C\left(C_{t+2}^*, t+2\right)\right]\right] \\ &= E_t\left[R_{it} R_{j,t+1} U_C\left(C_{t+2}^*, t+2\right)\right] \end{aligned} \tag{6.8}$$

or

$$1 = E_t \left[R_{it} R_{j,t+1} m_{t,t+2} \right] \tag{6.9}$$

where $m_{t,t+2} \equiv U_C\left(C^*_{t+2}, t+2\right)/U_C\left(C^*_t, t\right)$ is the marginal rate of substitution, or the stochastic discount factor, between dates t and $t+2$. In the preceding expressions, $R_{it}R_{j,t+1}$ is the return from a trading strategy that first invests in asset i over the period from t to $t+1$ then invests in asset j over the period $t+1$ to $t+2$. Of course, i could equal j but need not, in general. By repeated substitution, (6.9) can be generalized to

$$1 = E_t \left[\mathrm{R}_{t,t+k} m_{t,t+k} \right] \tag{6.10}$$

where $m_{t,t+k} \equiv U_C\left(C^*_{t+k}, t+k\right)/U_C\left(C^*_t, t\right)$ and $\mathrm{R}_{t,t+k}$ is the return from any trading strategy involving multiple assets over the period from dates t to $t+k$.

Equation (6.10) says that optimizing consumers equate their expected marginal utilities across all time periods and all states. Its equilibrium implication is that the stochastic discount relationship holds for multiperiod returns generated from any particular trading strategy. This result implies that empirical tests of multiperiod, time-separable utility models using consumption data and asset returns can be constructed using a wide variety of investment returns and holding periods. Expressions such as (6.10) represent moment conditions that are often tested using generalized method-of-moments techniques.[1] As mentioned in Chapter 4, such consumption-based tests typically reject models that assume standard forms of time-separable utility. This has motivated a search for alternative utility specifications, a topic we will revisit in future chapters.

Lets us now consider a general equilibrium structure for this multiperiod consumption-portfolio choice model.

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[1] See Lars Hansen and Kenneth Singleton (Hansen and Singleton 1983) and Lars Hansen and Ravi Jagannathan (Hansen and Jagannathan 1991).

6.2 The Lucas Model of Asset Pricing

The Lucas model (Lucas 1978) derives the equilibrium prices of risky assets for an *endowment* economy. An endowment economy is one where the random process generating the economy's real output (e.g., Gross Domestic Product, or GDP) is taken to be exogenous. Moreover, it is assumed that output obtained at a particular date cannot be reinvested to produce more output in the future. Rather, all output on a given date can only be consumed immediately, implying that equilibrium aggregate consumption equals the exogenous level of output at each date. Assets in this economy represent ownership claims on output, so that output (and consumption) on a given date can also be interpreted as the cash dividends paid to asset holders. Because reinvestment of output is not permitted, so that the scale of the production process is fixed, assets can be viewed as being perfectly inelastically supplied.[2]

As we will make explicit shortly, these endowment economy assumptions essentially fix the process for aggregate consumption. Along with the assumption that all individuals are identical, that is, that there is a representative individual, the endowment economy assumptions fix the processes for individuals' consumptions. Thus, individuals' marginal rates of substitution between current and future consumptions are pinned down, and the economy's stochastic discount factor becomes exogenous. Furthermore, since the exogenous output-consumption process also represents the process for the market portfolio's aggregate dividends, that too is exogenous. This makes it easy to solve for the equilibrium price of the market portfolio.

[2] An endowment economy is sometimes described as a "fruit tree" economy. The analogy refers to an economy whose production is represented by a fixed number of fruit trees. Each season (date), the trees produce a random amount of output in the form of perishable fruit. The only value to this fruit is to consume it immediately, as it cannot be reinvested to produce more fruit in the future. (Planting seeds from the fruit to increase the number of fruit trees is ruled out.) Assets represent ownership claims on the fixed number of fruit trees (orchards), so that the fruit produced on each date also equals the dividend paid to asset holders.

In contrast, a *production* economy is, in a sense, the polar opposite of an endowment economy. A production economy allows for an aggregate consumption-savings (investment) decision. Not all of current output need be consumed, but some can be physically invested to produce more output using constant returns to scale (linear) production technologies. The random distribution of rates of return on these productive technologies is assumed to be exogenous. Assets can be interpreted as ownership claims on these technological processes and, therefore, their supplies are perfectly elastic, varying in accordance to the individual's reinvestment decision. Hence, the main difference between production and endowment economies is that production economies pin down assets' rates of return distribution and make consumption (and output) endogenous, whereas endowment economies pin down consumption and make assets' rates of return distribution endogenous. Probably the best-known asset pricing model based on a production economy was derived by John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross (Cox, Ingersoll, and Ross 1985a). We will study this continuous-time, general equilibrium model in Chapter 13.

6.2.1 Including Dividends in Asset Returns

The Lucas model builds on the multiperiod, time-separable utility model of consumption and portfolio choice. We continue with the stochastic discount factor pricing relationship of equation (6.7) but put more structure on the returns of each asset. Let the return on the i^{th} risky asset, R_{it}, include a dividend payment made at date $t+1$, $d_{i,t+1}$, along with a capital gain, $P_{i,t+1} - P_{it}$. Hence, P_{it} denotes the ex-dividend price of the risky asset at date t:

$$R_{it} = \frac{d_{i,t+1} + P_{i,t+1}}{P_{it}} \tag{6.11}$$

Substituting (6.11) into (6.7) and rearranging gives

$$P_{it} = E_t \left[\frac{U_C\left(C^*_{t+1}, t+1\right)}{U_C\left(C^*_t, t\right)} \left(d_{i,t+1} + P_{i,t+1}\right) \right] \tag{6.12}$$

Similar to what was done in equation (6.8), if we substitute for $P_{i,t+1}$ using equation (6.12) updated one period, and use the properties of conditional expectation, we have

$$\begin{aligned} P_{it} &= E_t \left[\frac{U_C\left(C^*_{t+1}, t+1\right)}{U_C\left(C^*_t, t\right)} \quad d_{i,t+1} + \frac{U_C\left(C^*_{t+2}, t+2\right)}{U_C\left(C^*_{t+1}, t+1\right)} \left(d_{i,t+2} + P_{i,t+2}\right) \Big) \right] \\ &= E_t \left[\frac{U_C\left(C^*_{t+1}, t+1\right)}{U_C\left(C^*_t, t\right)} d_{i,t+1} + \frac{U_C\left(C^*_{t+2}, t+2\right)}{U_C\left(C^*_t, t\right)} \left(d_{i,t+2} + P_{i,t+2}\right) \right] \end{aligned} \tag{6.13}$$

Repeating this type of substitution, that is, solving forward the difference equation (6.13), gives us

$$P_{it} = E_t \left[\sum_{j=1}^{T} \frac{U_C\left(C^*_{t+j}, t+j\right)}{U_C\left(C^*_t, t\right)} d_{i,t+j} + \frac{U_C\left(C^*_{t+T}, t+T\right)}{U_C\left(C^*_t, t\right)} P_{i,t+T} \right] \tag{6.14}$$

where the integer T reflects a large number of future periods. Now suppose utility reflects a rate of time preference, so that $U\left(C_t, t\right) = \delta^t u\left(C_t\right)$, where $\delta = \frac{1}{1+\rho} < 1$, so that the rate of time preference $\rho > 0$. Then (6.14) becomes

$$P_{it} = E_t \left[\sum_{j=1}^{T} \delta^j \frac{u_C\left(C^*_{t+j}\right)}{u_C\left(C^*_t\right)} d_{i,t+j} + \delta^T \frac{u_C\left(C^*_{t+T}\right)}{u_C\left(C^*_t\right)} P_{i,t+T} \right] \tag{6.15}$$

If we have an infinitely lived individual or, equivalently, an individual whose utility includes a bequest that depends on the utility of his or her offspring, then we can consider the solution to (6.15) as the planning horizon, T, goes to infinity. If $\lim_{T \to \infty} E_t \left[\delta^T \frac{u_C\left(C^*_{t+T}\right)}{u_C\left(C^*_t\right)} P_{i,t+T} \right] = 0$, which (as discussed in the next section) is equivalent to assuming the absence of a speculative price "bubble," then

$$\begin{aligned} P_{it} &= E_t\left[\sum_{j=1}^{\infty}\delta^j \frac{u_C\left(C_{t+j}^*\right)}{u_C\left(C_t^*\right)} d_{i,t+j}\right] \\ &= E_t\left[\sum_{j=1}^{\infty} m_{t,\,t+j} d_{i,t+j}\right] \end{aligned} \tag{6.16}$$

Equation (6.16) is a present value formula, where the stochastic discount factors are the marginal rates of substitution between the present and the dates when the dividends are paid. This "discounted dividend" asset pricing formula holds for any individual following an optimal consumption-portfolio choice policy. Thus far, we have not made any strong assumptions about consumer homogeneity or the structure of the economy. For example, equation (6.16) would hold for a production economy with heterogeneous individuals.

6.2.2 Equating Dividends to Consumption

The Lucas model makes equation (6.16) into a general equilibrium model of asset pricing by assuming there is an infinitely lived *representative* individual, meaning that all individuals are identical with respect to utility and initial wealth. It also assumes that each asset is a claim on a real output process, where risky asset i pays a real dividend of d_{it} at date t. Moreover, the dividend from each asset is assumed to come in the form of a nonstorable consumption good that cannot be reinvested. In other words, this dividend output cannot be transformed into new investment in order to expand the scale of production. The only use for each asset's output is consumption. A share of risky asset i can be interpreted as an ownership claim on an exogenous dividend-output process that is fixed in supply. Assuming no wage income, it then follows that aggregate consumption at each date must equal the total dividends paid by all of the n assets at that date:

$$C_t^* = \sum_{i=1}^{n} d_{it} \tag{6.17}$$

Given the assumption of a representative individual, this individual's consumption can be equated to aggregate consumption.[3]

6.2.3 Asset Pricing Examples

With these endowment economy assumptions, the specific form of utility for the representative agent and the assumed distribution of the assets' dividend processes fully determine equilibrium asset prices. For example, if the representative individual is risk-neutral, so that u_C is a constant, then (6.16) becomes

$$P_{it} = E_t \left[\sum_{j=1}^{\infty} \delta^j d_{i,t+j} \right] \tag{6.18}$$

In words, the price of risky asset i is the expected value of dividends discounted by a constant factor, reflecting the constant rate of time preference.

Consider another example where utility is logarithmic, $u(C_t) = \ln C_t$. Also denote $d_t = \sum_{i=1}^{n} d_{it}$ to be the economy's aggregate dividends, which we know by (6.17) equals aggregate consumption. Then the price of risky asset i is given

[3] If one assumes that there are many representative individuals, each will have identical per capita consumption and receive identical per capita dividends. Hence, in (6.17), C_t^* and d_{it} can be interpreted as per capita quantities.

by

$$
\begin{aligned}
P_{it} &= E_t\left[\sum_{j=1}^{\infty}\delta^j \frac{C_t^*}{C_{t+j}^*} d_{i,t+j}\right] \\
&= E_t\left[\sum_{j=1}^{\infty}\delta^j \frac{d_t}{d_{t+j}} d_{i,t+j}\right] \qquad (6.19)
\end{aligned}
$$

Given assumptions regarding the distribution of the individual assets, the expectation in (6.19) can be computed. However, under this logarithmic utility assumption, we can obtain the price of the market portfolio of all assets even without any distributional assumptions. To see this, let P_t represent a claim on aggregate dividends. Then (6.19) becomes

$$
\begin{aligned}
P_t &= E_t\left[\sum_{j=1}^{\infty}\delta^j \frac{d_t}{d_{t+j}} d_{t+j}\right] \\
&= d_t \frac{\delta}{1-\delta} \qquad (6.20)
\end{aligned}
$$

implying that the value of the market portfolio moves in step with the current level of dividends. It does not depend on the distribution of future dividends. Why? Higher expected future dividends, d_{t+j}, are exactly offset by a lower expected marginal utility of consumption, $m_{t,\,t+j} = \delta^j d_t/d_{t+j}$, leaving the value of a claim on this output process unchanged. This is consistent with our earlier results showing that a log utility individual's savings (and consumption) are independent of the distribution of asset returns. Since aggregate savings equals the aggregate demand for the market portfolio, no change in savings implies no change in asset demand. Note that this will not be the case for the more general specification of power (constant relative-risk-aversion) utility. If $u(C_t) = C_t^{\gamma}/\gamma$, then

$$\begin{aligned} P_t &= E_t \left[\sum_{j=1}^{\infty} \delta^j \left(\frac{d_{t+j}}{d_t} \right)^{\gamma-1} d_{t+j} \right] \\ &= d_t^{1-\gamma} E_t \left[\sum_{j=1}^{\infty} \delta^j d_{t+j}^{\gamma} \right] \end{aligned} \tag{6.21}$$

which does depend on the distribution of future aggregate dividends (output). Note from (6.21) that for the case of certainty ($E_t \left[d_{t+j}^{\gamma} \right] = d_{t+j}^{\gamma}$), when $\gamma < 0$ higher future aggregate dividends reduce the value of the market portfolio, that is, $\partial P_t / \partial d_{t+j} = \gamma \delta^j \left(d_{t+j}/d_t \right)^{\gamma-1} < 0$. While this seems counterintuitive, recall that for $\gamma < 0$, individuals desire less savings (and more current consumption)

when investment opportunities improve. Since current consumption is fixed at d_t in this endowment economy, the only way to bring higher desired consumption back down to d_t is for total wealth to decrease. In equilibrium, this occurs when the price of the market portfolio falls as individuals attempt to sell some of their portfolio in an (unsuccessful) attempt to raise consumption. Of course, the reverse story occurs when $0 < \gamma < 1$, as a desired rise in savings is offset by an increase in wealth via an appreciation of the market portfolio.

If we continue to assume power utility, we can also derive the value of a hypothetical riskless asset that pays a one-period dividend of \$1:

$$P_{ft} = \frac{1}{R_{ft}} = \delta E_t \left[\left(\frac{d_{t+1}}{d_t} \right)^{\gamma-1} \right] \tag{6.22}$$

Using aggregate U.S. consumption data, Rajnish Mehra and Edward C. Prescott (Mehra and Prescott 1985) used equations such as (6.21) and (6.22) with $d_t = C_t^*$ to see if a reasonable value of γ would produce a risk premium (excess average return over a risk-free return) for a market portfolio of U.S. common stocks that matched these stocks' historical average excess returns. They found that for reasonable values of γ, they could not come close to the historical risk premium, which at that time they estimated to be around 6 percent. They described this finding as the *equity premium puzzle*. As mentioned in Chapter 4, the problem is that for reasonable levels of risk aversion, aggregate consumption appears to vary too little to justify the high Sharpe ratio for the market portfolio of stocks. The moment conditions in (6.21) and (6.22) require a highly negative value of γ to fit the data.

6.2.4 A Lucas Model with Labor Income

The Lucas endowment economy model has been modified to study a wide array of issues. For example, Gurdip Bakshi and Zhiwu Chen (Bakshi and Chen 1996) studied a monetary endowment economy by assuming that a representative individual obtains utility from both real consumption and real money balances. In future chapters, we will present other examples of Lucas-type economies where utility is non-time-separable and where utility reflects psychological biases. In this section, we present a simplified version of a model by Stephen Cecchetti, Pok-sang Lam, and Nelson Mark (Cecchetti, Lam, and Mark 1993) that modifies the Lucas model to consider nontraded labor income.[4]

As before, suppose that there is a representative agent whose financial wealth consists of a market portfolio of traded assets that pays an aggregate real dividend of d_t at date t. We continue to assume that these assets are in fixed supply and their dividend consists of a nonstorable consumption good. However, now we also permit each individual to be endowed with nontradeable human capital

[4] They use a regime-switching version of this model to analyze the equity premium and risk-free rate puzzles. Based on Generalized Method of Moments (GMM) tests, they find that their model fits the first moments of the risk-free rate and the return to equity, but not the second moments.

that is fixed in supply. The agent's return to human capital consists of a wage payment of y_t at date t that also takes the form of the nonstorable consumption good. Hence, equilibrium per capita consumption will equal

$$C_t^* = d_t + y_t \tag{6.23}$$

so that it is no longer the case that equilibrium consumption equals dividends. However, assuming constant relative-risk-aversion utility, the value of the market portfolio can still be written in terms of future consumption and dividends:

$$\begin{aligned} P_t &= E_t\left[\sum_{j=1}^{\infty}\delta^j \frac{u_C\left(C_{t+j}^*\right)}{u_C\left(C_t^*\right)} d_{t+j}\right] \\ &= E_t\left[\sum_{j=1}^{\infty}\delta^j \left(\frac{C_{t+j}^*}{C_t^*}\right)^{\gamma-1} d_{t+j}\right] \end{aligned} \tag{6.24}$$

Because wage income creates a difference between aggregate dividends and equilibrium consumption, its presence allows us to assume separate random processes for dividends and consumption. For example, one might assume dividends and equilibrium consumption follow the lognormal processes:

$$\begin{aligned} \ln\left(C_{t+1}^*/C_t^*\right) &= \mu_c + \sigma_c \eta_{t+1} \\ \ln\left(d_{t+1}/d_t\right) &= \mu_d + \sigma_d \varepsilon_{t+1} \end{aligned} \tag{6.25}$$

where the error terms are serially uncorrelated and distributed as

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \tilde{} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \tag{6.26}$$

It is left as an end-of-chapter exercise to show that with these assumptions regarding the distributions of C^*_{t+j} and d_{t+j}, when $\delta e^{\alpha} < 1$ one can compute the expectation in (6.24) to be

$$P_t = d_t \frac{\delta e^{\alpha}}{1 - \delta e^{\alpha}} \tag{6.27}$$

where

$$\alpha \equiv \mu_d - (1-\gamma)\,\mu_c + \frac{1}{2}\left[(1-\gamma)^2\,\sigma_c^2 + \sigma_d^2\right] - (1-\gamma)\,\rho\sigma_c\sigma_d \tag{6.28}$$

We can confirm that (6.27) equals (6.20) when $\gamma = 0$, $\mu_d = \mu_c$, $\sigma_c = \sigma_d$, and $\rho = 1$, which is the special case of log utility and no labor income. With no labor income ($\mu_d = \mu_c$, $\sigma_c = \sigma_d$, $\rho = 1$) but $\gamma \neq 0$, we have $\alpha = \gamma\mu_c + \frac{1}{2}\gamma^2\sigma_c^2$, which is increasing in the growth rate of dividends (and consumption) when $\gamma > 0$. As discussed in Chapter 4, this occurs because greater dividend growth leads individuals to desire increased savings since they have high intertemporal elasticity ($\varepsilon = 1/(1-\gamma) > 1$). An increase in desired savings reflects the substitution effect exceeding the income or wealth effect. Market clearing then requires the value of the market portfolio to rise, raising income or wealth to make desired consumption rise to equal the fixed supply. The reverse occurs when $\gamma < 0$, as the income or wealth effect will exceed the substitution effect.

For the general case of labor income where α is given by equation (6.28), note that a lower correlation between consumption and dividends (decline in ρ) increases α. Since $\partial P_t/\partial\alpha > 0$, this lower correlation raises the value of the market portfolio. Intuitively, this greater demand for the market portfolio results because it provides better diversification with uncertain labor income.

6.3 Rational Asset Price Bubbles

In this section we examine whether there are solutions other than (6.16) that can satisfy the asset price difference equation (6.15). Indeed, we will show that there are and that these alternative solutions can be interpreted as *bubble* solutions where the asset price deviates from its fundamental value. Potentially, these bubble solutions may be of interest because there appear to be numerous historical episodes during which movements in asset prices appear inconsistent with reasonable dynamics for dividends or outputs. In other words, assets do not appear to be valued according to their fundamentals. Examples include the Dutch tulip bulb bubble during the 1620s, the Japanese stock price bubble during the late 1980s, and the U.S. stock price bubble (particularly Internet-related stocks) during the late 1990s.[5] While some may conclude that these bubbles represent direct evidence of irrational behavior on the part of individual investors, might an argument be made that bubbles could be consistent with rational actions and beliefs? It is this possibility that we now consider.[6]

Let us start by defining $p_t \equiv P_{it} u_C\left(C_t\right)$ as the product of the asset price and the marginal utility of consumption, excluding the time preference discount factor, δ. Then equation (6.12) can be written as the difference equation:

$$E_t\left[p_{t+1}\right] = \delta^{-1} p_t - E_t\left[u_C\left(C_{t+1}^*\right) d_{i,t+1}\right] \tag{6.29}$$

where $\delta^{-1} = 1 + \rho > 1$ with ρ being the individual's subjective rate of time preference. The solution (6.16) to this equation is referred to as the *fundamental* solution. Let us denote it as f_t:

[5] Charles P. Kindleberger (Kindleberger 2001) gives an entertaining account of numerous asset price bubbles.

[6] Of course, another possibility is that asset prices always equal their fundamental values, and sudden rises and falls in these prices reflect sudden changes in perceived fundamentals.

$$p_t = f_t \equiv E_t \left[\sum_{j=1}^{\infty} \delta^j u_C \left(C^*_{t+j} \right) d_{i,t+j} \right] \tag{6.30}$$

The sum in (6.30) converges as long as the marginal utility-weighted dividends are expected to grow more slowly than the time preference discount factor. For the Lucas endowment economy, assumptions regarding the form of utility and the distribution of the assets' dividends can ensure that this solution has a finite value.

While f_t satisfies (6.29), it is not the only solution. Solutions that satisfy (6.29) take the general form $p_t = f_t + b_t$, where the *bubble* component of the solution is any process that satisfies

$$E_t \left[b_{t+1} \right] = \delta^{-1} b_t \tag{6.31}$$

This is easily verified by substitution into (6.29):

$$\begin{aligned}
E_t \left[f_{t+1} + b_{t+1} \right] &= \delta^{-1} \left(f_t + b_t \right) - E_t \left[u_C \left(C^*_{t+1} \right) d_{i,t+1} \right] \\
E_t \left[f_{t+1} \right] + E_t \left[b_{t+1} \right] &= \delta^{-1} f_t + \delta^{-1} b_t - E_t \left[u_C \left(C^*_{t+1} \right) d_{i,t+1} \right] \\
E_t \left[b_{t+1} \right] &= \delta^{-1} b_t
\end{aligned} \tag{6.32}$$

where in the last line of (6.32), we use the fact that f_t satisfies the difference equation. Note that since $\delta^{-1} > 1$, b_t explodes in expected value:

$$\lim_{i \to \infty} E_t \left[b_{t+i} \right] = \lim_{i \to \infty} \delta^{-i} b_t = \begin{cases} +\infty \text{ if } b_t > 0 \\ -\infty \text{ if } b_t < 0 \end{cases} \tag{6.33}$$

The exploding nature of b_t provides a rationale for interpreting the general solution $p_t = f_t + b_t$, $b_t \neq 0$, as a bubble solution. Only when $b_t = 0$ do we get the fundamental solution.

6.3.1 Examples of Bubble Solutions

Suppose that b_t follows a deterministic time trend; that is,

$$b_t = b_0 \delta^{-t} \tag{6.34}$$

Then the solution

$$p_t = f_t + b_0 \delta^{-t} \tag{6.35}$$

implies that the marginal utility-weighted asset price grows exponentially forever. In other words, we have an ever-expanding speculative bubble.

Next, consider a possibly more realistic modeling of a "bursting" bubble proposed by Olivier Blanchard (Blanchard 1979):

$$b_{t+1} = \begin{cases} (\delta q)^{-1} b_t + e_{t+1} & \text{with probability } q \\ z_{t+1} & \text{with probability } 1-q \end{cases} \tag{6.36}$$

with $E_t[e_{t+1}] = E_t[z_{t+1}] = 0$. Note that this process satisfies the condition in (6.31), so that $p_t = f_t + b_t$ is again a valid bubble solution. In this case, the bubble continues with probability q each period but "bursts" with probability $1-q$. If it bursts, it returns in expected value to zero, but then a new bubble would start. To compensate for the probability of a "crash," the expected return, conditional on not crashing, is higher than in the previous example of a never-ending bubble. The disturbance e_t allows bubbles to have additional noise and allows new bubbles to begin after the previous bubble has crashed. This bursting bubble model can be generalized to allow q to be stochastic.[7]

[7] The reader is asked to show this in an exercise at the end of the chapter.

6.3.2 The Likelihood of Rational Bubbles

While these examples of bubble solutions indeed satisfy the asset pricing difference equation in (6.29), there may be additional economic considerations that rule them out. One issue involves negative bubbles, that is, cases where $b_t < 0$. From (6.33) we see that individuals must expect that, at some future date $\tau > t$, the marginal utility-weighted price $p_\tau = f_\tau + b_\tau$ will become negative. Of course, since marginal utility is always positive, this implies that the asset price, $P_{it} = p_t/u_C(C_t)$, will also be negative. A negative price would be inconsistent with limited-liability securities, such as typical shareholders' equity (stocks). Moreover, if an individual can freely dispose of an asset, its price cannot be negative. Hence, negative bubbles can be ruled out.

Based on similar reasoning, Behzad Diba and Herschel Grossman (Diba and Grossman 1988) argue that many types of bubble processes, including bubbles that burst and start again, can also be ruled out. Their argument is as follows. Note that the general process for a bubble can be written as

$$b_t = \delta^{-t} b_0 + \sum_{s=1}^{t} \delta^{s-t} \varepsilon_s \tag{6.37}$$

where ε_s, $s = 1, ..., t$ are mean-zero innovations. To avoid negative values of b_t (and negative expected future prices), realizations of ε_t must satisfy

$$\varepsilon_t \geq -\delta^{-1} b_{t-1}, \ \ \forall t \geq 0 \tag{6.38}$$

For example, suppose that $b_t = 0$, implying that, at the current date t, a bubble does not exist. Then from (6.38) and the requirement that ε_{t+1} have mean zero, it must be the case that $\varepsilon_{t+1} = 0$ with probability 1. This implies that if a bubble currently does not exist, it cannot get started next period or at any future period. The only possibility would be if a positive bubble existed on the first day of trading of the asset; that is, $b_0 > 0$.[8] Moreover, the bursting and then restarting bubble in (6.36) could only avoid a negative value of b_{t+1} if $z_{t+1} = 0$ with probability 1 and $e_{t+1} = 0$ whenever $b_t = 0$. Hence, this type of bubble would need to be positive on the first trading day, and once it bursts it could never restart.

Note, however, that arbitrage trading is unlikely to be a strong argument against a bursting bubble. While short-selling an asset with $b_t > 0$ would result in a profit when the bubble bursts, the short-seller could incur substantial losses beforehand. Over the near term, if the bubble continues, the market value of the short-seller's position could become sufficiently negative so as to wipe out his personal wealth.

Other arguments have been used to rule out positive bubbles. Similar to the assumptions underlying the Lucas model of the previous section, Jean Tirole (Tirole 1982) considers a situation with a finite number of rational individuals and where the dividend processes for risky assets are exogenously given. In such an economy, individuals who trade assets at other than their fundamental prices are playing a zero-sum game, since the aggregate amounts of consumption and wealth are exogenous. Trading assets at prices having a bubble component only transfers claims on this fixed supply of wealth between individuals. Hence, a rational individual will not purchase an asset whose price already reflects a positive bubble component. This is because at a positive price, previous traders in the asset have already realized their gains and left a negative-sum game to the subsequent traders. The notion that an individual would believe that he can buy an asset at a positive bubble price and later sell it to another at a price reflecting an even greater bubble component might be considered a "greater

[8] An implication is that an initial public offering (IPO) of stock should have a first-day market price that is above its fundamental value. Interestingly, Jay Ritter (Ritter 1991) documents that many IPOs initially appear to be overpriced since their subsequent returns tend to be lower than comparable stocks.

fool" theory of speculative bubbles. However, this theory is not consistent with a finite number of fully rational individuals in most economic settings. Manual Santos and Michael Woodford (Santos and Woodford 1997) consider the possibility of speculative bubbles in a wide variety of economies, including those with overlapping generations of individuals. They conclude that the conditions necessary for rational speculative bubbles to exist are relatively fragile. Under fairly general assumptions, equilibria displaying rational price bubbles can be excluded.[9]

6.4 Summary

When individuals choose lifetime consumption and portfolio holdings in an optimal fashion, a multiperiod stochastic discount factor can be used to price assets. This is an important generalization of our earlier single-period pricing result. We also demonstrated that if an asset's dividends (cashflows) are modeled explicitly, the asset's price satisfies a discounted dividend formula. The Lucas endowment economy model took this discounted dividend formula a step further by equating aggregate dividends to aggregate consumption. This simplified valuing a claim on aggregate dividends, since now the value of this market portfolio could be expressed as an expectation of a function of only the future dividend (output) process.

In an infinite horizon model, the possibility of rational asset price bubbles needs to be considered. In general, there are multiple solutions for the price of a risky asset. Bubble solutions represent nonstationary alternatives to the asset's fundamental value. However, when additional aspects of the economic envi-

[9] Of course, other considerations that are not fully consistent with rationality may give rise to bubbles. José Scheinkman and Wei Xiong (Scheinkman and Xiong 2003) present a model where individuals with heterogeneous beliefs think that particular information is more informative of asset fundamentals than it truly is. Bubbles arise due to a premium reflecting the option to sell assets to the more optimistic individuals.

ronment are considered, the conditions that would give rise to rational bubbles appear to be rare.

6.5 Exercises

1. Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02}f_2]$? Explain your answer.

2. Assume that there is an economy populated by infinitely lived representative individuals who maximize the lifetime utility function

$$E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t}\right]$$

where c_t is consumption at date t and $a > 0$, $0 < \delta < 1$. The economy is a Lucas endowment economy (Lucas 1978) having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

3. For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship of equations (6.27) and (6.28).

4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely lived investors are risk-

neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t that is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty} \frac{E_t\left[d_{t+i}\right]}{{R_f}^{i+1}} \tag{1}$$

and

$$b_{t+1} = \begin{cases} \frac{R_f}{q_t} b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1 - q_t \end{cases} \tag{2}$$

where $E_t\left[e_{t+1}\right] = E_t\left[z_{t+1}\right] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

a. Show whether or not $p_t = f_t + b_t$, subject to the specifications in (1) and (2), is a valid solution for the price of the risky asset.

b. Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{solar}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

c. Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.

5. Consider an endowment economy with representative agents who maximize the following objective function:

$$\max_{C_s,\{\omega_{is}\},\forall s,i} E_t\left[\sum_{s=t}^{T}\delta^s u\left(C_s\right)\right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.