# Limits to Arbitrage

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Asset Pricing Theory

# I.Example: CARA Utility and Normal Asset Returns

- Several single-period portfolio choice models assume constant absolute risk-aversion (CARA) utility and normally distributed asset returns due to the analytical convenience of these assumptions.
- CARA utility takes the negative exponential form $U(C) = -e^{-bC}$, $b > 0$.
- As before, let $W_0$ and $C_0$ be initial wealth and consumption, and let $C_1$ be end-of-period consumption.
- Let there be a risk-free asset with return $R_f$ and $n$ risky assets with the $n \times 1$ vector of random returns $\tilde{R} \sim N\left(\overline{R}, V\right)$ where $\overline{R}$ is the $n \times 1$ vector of expected returns and $V$ is the $n \times n$ matrix of return covariances.

# Maximization Problem

- Let $\omega = (\omega_1 \ldots \omega_n)'$ and $\mathbf{1}$ be $n \times 1$ vectors of risky asset portfolio weights and ones. Assuming no labor income, then

$$C_1 = (W_0 - C_0)\left[R_f + \omega'(\tilde{R} - R_f\mathbf{1})\right] \tag{1}$$

- The individual's maximization problem is

$$\max_{C_0,\omega} - e^{-bC_0} + \delta E\left[-e^{-b(W_0-C_0)\left[R_f+\omega'(\tilde{R}-R_f\mathbf{1})\right]}\right] \tag{2}$$

- Since $R_f + \omega'(\tilde{R} - R_f\mathbf{1})$ is normally distributed, (2) equals[1]

$$\max_{C_0,\omega} - e^{-bC_0} - \delta e^{-b(W_0-C_0)[R_f+\omega'(\bar{R}-R_f\mathbf{1})]+\frac{1}{2}b^2(W_0-C_0)^2\omega' V\omega} \tag{3}$$

[1] If $x \sim N(\mu, \sigma^2)$, then $\exp(x)$ is lognormally distributed and $E[\exp(x)] = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$.

# CARA-Normal Portfolio Choice

- If we first consider only the individual's choice of risky asset portfolio weights, note that the maximization problem (3) with respect to $\omega$ is equivalent to

$$\max_{\omega} \omega'(\bar{R} - R_f \mathbf{1}) - \frac{1}{2} b (W_0 - C_0) \omega' V \omega \tag{4}$$

- In vector notation, the $n$ first-order conditions are

$$\bar{R} - R_f \mathbf{1} - b (W_0 - C_0) V \omega = 0 \tag{5}$$

# CARA-Normal Portfolio Choice

- Solving for the amount of savings invested the risky assets:

$$\omega^* (W_0 - C_0) = \frac{1}{b} V^{-1} (\bar{R} - R_f \mathbf{1}) \tag{6}$$

- Note that the amount invested in the risky assets decreases with absolute risk-aversion, $b$.

- However, this CARA utility individual invests a fixed amount in the risky assets, independent of initial savings or wealth.

- The amount invested in the risk-free asset is $(1 - \omega' \mathbf{1}) (W_0 - C_0)$, which increases one-for-one with an increase in saving.

# CARA-Normal Consumption Choice

- Since from (6) the risky asset investments are independent of wealth or initial consumption (and savings), (3) simplifies to

$$\max_{C_0} -e^{-bC_0} - \delta e^{-b(W_0-C_0)R_f - \frac{1}{2}(\bar{R}-R_f\mathbf{1})'V^{-1}(\bar{R}-R_f\mathbf{1})} \tag{7}$$

- The first order condition with respect to $C_0$ is

$$be^{-bC_0} - bR_f\delta e^{-b(W_0-C_0)R_f - \frac{1}{2}(\bar{R}-R_f\mathbf{1})'V^{-1}(\bar{R}-R_f\mathbf{1})} = 0$$

Dividing by $b$ and taking logs:

$$-bC_0 = \ln(R_f\delta) - b(W_0 - C_0)R_f - \frac{1}{2}(\bar{R} - R_f\mathbf{1})'V^{-1}(\bar{R} - R_f\mathbf{1})$$

which implies

$$C_0^* = \frac{W_0R_f}{1+R_f} - \frac{\ln(R_f\delta) - \frac{1}{2}(\bar{R} - R_f\mathbf{1})'V^{-1}(\bar{R} - R_f\mathbf{1})}{b(1+R_f)} \tag{8}$$

## 2. Limits to Arbitrage

- In (6) we solved for a CARA investor's optimal demands for $n$ normally-distributed risky assets:

$$\omega^* (W_0 - C_0) = \frac{1}{b} V^{-1} (\bar{R} - R_f \mathbf{1}) \tag{9}$$

- Consider the case of two risky assets, Assets A and B where

$$V = \begin{pmatrix} \sigma_A^2 & \rho\sigma_A\sigma_B \\ \rho\sigma_A\sigma_B & \sigma_B^2 \end{pmatrix} \tag{10}$$

and

$$\bar{R} = \begin{pmatrix} \bar{R}_A \\ \bar{R}_B \end{pmatrix} = \begin{pmatrix} \bar{X}_A / P_A \\ \bar{X}_B / P_B \end{pmatrix} \tag{11}$$

- Equation (11) shows that expected returns, $\bar{R}_i$, $i = A, B$ equal the end-of-period expected payoff or dividend, $\bar{X}_i$, divided by the initial price, $P_i$.

# Asset Supplies

- Define $(w_A \;\; w_B)' \equiv (W_0 - C_0)(\omega_A \;\; \omega_B)'$ as the initial amounts demanded for the risky assets. Then (9) is:

$$\begin{pmatrix} w_A \\ w_B \end{pmatrix} = \frac{1}{b\left(1-\rho^2\right)} \begin{pmatrix} \frac{\bar{R}_A - R_f}{\sigma_A^2} - \frac{\rho(\bar{R}_B - R_f)}{\sigma_A \sigma_B} \\ \frac{\bar{R}_B - R_f}{\sigma_B^2} - \frac{\rho(\bar{R}_A - R_f)}{\sigma_A \sigma_B} \end{pmatrix} \qquad (12)$$

- Gromb and Vayanos (2010) implicitly assume that the supplies of Asset B and the risk-free asset are perfectly elastic, which may be justified by a production economy similar to Cox, Ingersoll, and Ross (1985) where constant returns to scale technologies determine assets' return processes.
- Thus, it is assumed that $\bar{R}_B = R_f$ irrespective of the demand for these assets.
- In contrast, Asset A's supply is assumed to be fixed at zero.

# Arbitrageur and Liquidity Provision

- Gromb and Vayanos (2010) study a limited arbitrage setting. They consider the model investor to be an arbitrageur.
- There are assumed to be other "outside" investors whose total net demand for Asset A is simply an exogenous amount $u$.
- The demand "shock" $u$ means that the total demand for Asset A is $u + w_A$.
- Since supply equals zero, it must be that $w_A = -u$. In this sense, the arbitrageur provides liquidity to the market for Asset A.

# Market Clearing

- Of course the arbitrageur must be induced to take the opposite side of the demand shock because there really is not a true arbitrage unless $\rho^2 = 1$.
- This occurs by an adjustment of the equilibrium rate of return, $\bar{R}_A = \bar{X}_A / P_A$.
- Given that the expected end-of-period dividend is fixed, adjustment implies that Asset A's initial price, $P_A$, adjusts to clear the market.

# Equilibrium Price

- With the assumptions that $\bar{R}_B = R_f$ and $w_A = -u$, from (12) the equilibrium price is

$$P_A = \frac{\bar{X}_A}{R_f - b\sigma_A^2 \left(1 - \rho^2\right) u} \tag{13}$$

- Consequently, a positive (*negative*) demand shock raises (*lowers*) the initial price of Asset A and lowers (*raises*) its expected rate of return $\bar{R}_A = \bar{X}_A / P_A$.
- Since from (13) $\bar{R}_A = \bar{X}_A / P_A = R_f - b\sigma_A^2 \left(1 - \rho^2\right) u < R_f$ when $u > 0$, we see from (12) that the arbitrageur is induced to (short) sell Asset A.

# Price Impact of Demand Shock

- Since

$$\frac{\partial P_A}{\partial u} = \frac{\bar{X}_A b\sigma_A^2 \left(1-\rho^2\right)}{\left(R_f - b\sigma_A^2 \left(1-\rho^2\right) u\right)^2} = P_A \frac{b\sigma_A^2 \left(1-\rho^2\right)}{R_f - b\sigma_A^2 \left(1-\rho^2\right) u} \quad , \tag{14}$$

  the impact of a demand shock is greater the

  1. greater is the arbitrageur's risk aversion, $b$.
  2. greater is the Asset A's volatility, $\sigma_A$.
  3. less perfect is hedging with Asset B, $\left(1-\rho^2\right)$.

- Thus, arbitrageur risk aversion, asset risk, and the absence of perfect hedging limit pure arbitrage and make Asset A's price deviate from its "fundamental" price of $\bar{X}_A / R_f$.

# Short Sale Constraints

- A cost to short sell Asset A might be modeled as reducing the arbitrageur's return by a proportional amount per share, $c$, whenever $w_A < 0$.
- Assuming as before that $\bar{R}_B = R_f$, then similar to (4) the arbitrageur's maximization problem is

$$\max_{\omega_A, \omega_B} \ \omega_A(\bar{R}_A - R_f) - (c/P_A)\left|\omega_A\right| 1_{\{\omega_A < 0\}} - \frac{b}{2}(W_0 - C_0)\left[\omega_A^2\sigma_A^2 + \omega_B^2\sigma_B^2 + 2\omega_A\omega_B\rho\sigma_A\sigma_B\right] \tag{15}$$

# Equilibrium Prices with Short Sale Costs

- Evaluating the first order conditions at the market clearing condition $w_A \equiv (W_0 - C_0)\,\omega_A = -u$ leads to

$$P_A = \begin{cases} \bar{X}_A / \left[R_f - b\sigma_A^2\left(1-\rho^2\right)u\right] & \text{if } u \leq 0 \\ \left(\bar{X}_A + c\right) / \left[R_f - b\sigma_A^2\left(1-\rho^2\right)u\right] & \text{if } u > 0 \end{cases} \quad (16)$$

- Compared to (13), the price of Asset A is higher by $c/\left[R_f - b\sigma_A^2\left(1-\rho^2\right)u\right]$ when there is a positive demand shock.

- The higher price is needed to compensate arbitrageurs for the cost of short-selling.

- Note that even if $\rho^2 = 1$ so that arbitrage would be perfect, short selling costs lead to a deviation from the Law of One Price.