Behavioral Finance and Asset Pricing

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Introduction

- We present models of asset pricing where investors' preferences are subject to psychological biases or where investors make systematic errors in judging the probability distribution of asset returns.
- A model that incorporates some form of irrationality attempts to provide a positive or descriptive theory of individual behavior (behavioral finance).
- We first consider Barberis, Huang, and Santos' (2001) model of an endowment economy where investors' decisions exhibit prospect theory.
- Second, we examine the model of Kogan, Ross, Wang, and Westerfield (2006) where some investors suffer systematic optimism or pessimism.

Prospect Theory

- *Prospect Theory* (Kahneman and Tversky (1979)) specifies investor utility that is a function of recent changes in, rather than simply the current level of, financial wealth.
- An example is *loss aversion* which characterizes investor utility that is more sensitive to recent losses than recent gains in financial wealth.
- A related bias is the *house money* effect which characterizes utility where losses following previous losses create more disutility than losses following previous gains: After a run-up in asset prices, the investor is less risk averse because subsequent losses would be "cushioned" by the previous gains.

Implications of Prospect Theory

- As shown by the Barberis, Huang, and Santos model, loss aversion together with the house money effect have implications for the dynamics of asset prices.
- After a substantial rise in asset prices, lower investor risk aversion can drive prices even higher, making asset price volatility exceed that of fundamentals (dividends).
- These biases also generate predictability in asset returns since a substantial recent fall (*rise*) in asset prices increases (*decreases*) risk aversion and expected asset returns.
- These biases can also imply a high equity risk premium because the "excess" volatility in equity prices leads loss-averse investors to demand a relatively high average rate of return on equities.

Barberis, Huang, Santos Model Assumptions

- **Technology**: There is a discrete-time endowment economy where the risky asset portfolio pays a date t perishable dividend of D_t. Date t aggregate consumption, $\overline{C}_t$, equals this dividend, D_t, plus perishable nonfinancial income, Y_t.

- $\overline{C}_t$ and D_t, follow the joint lognormal process

$$\begin{aligned} \ln\left(\overline{C}_{t+1}/\overline{C}_t\right) &= g_C + \sigma_C \eta_{t+1} \\ \ln\left(D_{t+1}/D_t\right) &= g_D + \sigma_D \varepsilon_{t+1} \end{aligned} \qquad (1)$$

where η_{t+1} and ε_{t+1} are serially uncorrelated and distributed

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \qquad (2)$$

Assumptions (continued)

- Let the risky asset return from date t to date $t+1$ be $R_{t+1} \equiv (P_{t+1} + D_{t+1}) / P_t$, and let the zero-net supply risk-free asset return from t to $t+1$ be $R_{f,t}$.

- **Preferences**: Representative, infinitely lived individuals maximize

$$E_0 \left[\sum_{t=0}^{\infty} \left(\delta^t \frac{C_t^{\gamma}}{\gamma} + b_t \delta^{t+1} v \left(X_{t+1}, w_t, z_t \right) \right) \right] \tag{3}$$

  where C_t is the individual's consumption.

- w_t is the amount of the risky asset held by the individual at date t.

Assumptions (continued)

- X_{t+1} is the total excess return earned on the risky asset from t to $t+1$ and is defined as

$$X_{t+1} \equiv w_t \left(R_{t+1} - R_{f,t} \right) \tag{4}$$

- $z_t < (>) \, 1$ measures prior gains (*losses*) on the risky asset:

$$z_t = (1 - \eta) + \eta z_{t-1} \frac{\overline{R}}{R_t} \tag{5}$$

where $0 \leq \eta \leq 1$ and $\overline{R}$ is a parameter, approximately equal to the average risky-asset return. The greater is η, the longer is the investor's memory in measuring gains from the risky asset.

Assumptions (continued)

- $v(\cdot)$ models prospect theory's effect of risky-asset gains/losses.
- If $z_t = 1$ (no prior gains/losses), $v(\cdot)$ displays pure loss aversion:

$$v(X_{t+1}, w_t, 1) = \begin{cases} X_{t+1} & \text{if } X_{t+1} \geq 0 \\ \lambda X_{t+1} & \text{if } X_{t+1} < 0 \end{cases} \tag{6}$$

where $\lambda > 1$. If $z_t \neq 1$, $v(\cdot)$ reflects the house money effect. For prior gains ($z_t \leq 1$), it equals

$$\begin{aligned} & v(X_{t+1}, w_t, z_t) \\ = & \begin{cases} X_{t+1} & \text{if } R_{t+1} \geq z_t R_{f,t} \\ X_{t+1} + (\lambda - 1) w_t (R_{t+1} - z_t R_{f,t}) & \text{if } R_{t+1} < z_t R_{f,t} \end{cases} \end{aligned} \tag{7}$$

Assumptions (continued)

- For prior losses ($z_t > 1$), it equals

$$v\left(X_{t+1}, w_t, z_t\right) = \begin{cases} X_{t+1} & \text{if } X_{t+1} \geq 0 \\ \lambda\left(z_t\right) X_{t+1} & \text{if } X_{t+1} < 0 \end{cases} \tag{8}$$

  where $\lambda\left(z_t\right) = \lambda + k\left(z_t - 1\right)$, $k > 0$. Losses that follow previous losses are penalized at the rate of $\lambda\left(z_t\right)$, which exceeds λ.

- The prospect theory term in the utility function is scaled to make the risky asset price-dividend ratio and the risky asset risk premium stationary with increases in aggregate wealth:

$$b_t = b_0 \overline{C}_t^{\gamma - 1} \tag{9}$$

  where $b_0 > 0$.

Solving the Model

- The state variables for the individual's consumption-portfolio choice problem are wealth, W_t, and z_t. We assume $f_t \equiv P_t / D_t = f_t(z_t)$ and then show that an equilibrium exists in which this is true. Hence, the return on the risky asset can be written

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{1 + f(z_{t+1})}{f(z_t)} \frac{D_{t+1}}{D_t} = \frac{1 + f(z_{t+1})}{f(z_t)} e^{g_D + \sigma_D \varepsilon_{t+1}} \tag{10}$$

- Let $R_{f,t} = R_f$, a constant, which will be verified by the solution to the agent's first-order conditions. Making this assumption simplifies the form of $v(\cdot)$.

Solution (continued)

- Note from (7) and (8) that $v\left(X_{t+1}, w_t, z_t\right)$ can be written $v\left(X_{t+1}, w_t, z_t\right) = w_t \widehat{v}\left(R_{t+1}, z_t\right)$, where for $z_t < 1$

$$\begin{aligned} & \widehat{v}\left(R_{t+1}, z_t\right) \\ = & \begin{cases} R_{t+1} - R_f & \text{if } R_{t+1} \geq z_t R_f \\ R_{t+1} - R_f + (\lambda - 1)\left(R_{t+1} - z_t R_f\right) & \text{if } R_{t+1} < z_t R_f \end{cases} \end{aligned} \tag{11}$$

and for $z_t > 1$

$$\widehat{v}\left(R_{t+1}, z_t\right) = \begin{cases} R_{t+1} - R_f & \text{if } R_{t+1} \geq R_f \\ \lambda\left(z_t\right)\left(R_{t+1} - R_f\right) & \text{if } R_{t+1} < R_f \end{cases} \tag{12}$$

Solution (continued)

- The individual's maximization problem is then

$$\max_{\{C_t, w_t\}} E_0 \left[\sum_{t=0}^{\infty} \left(\delta^t \frac{C_t^\gamma}{\gamma} + b_0 \delta^{t+1} \overline{C}_t^{\gamma-1} w_t \widehat{v} \left(R_{t+1}, z_t \right) \right) \right] \quad (13)$$

subject to the budget constraint

$$W_{t+1} = \left(W_t + Y_t - C_t \right) R_f + w_t \left(R_{t+1} - R_f \right) \quad (14)$$

and the dynamics for z_t given in (5).

Derived Utility of Wealth

- Define $\delta^t J(W_t, z_t)$ as the derived utility-of-wealth function.

- Then the Bellman equation for this problem is

$$
\begin{aligned}
J(W_t, z_t) \quad = \quad & \max_{\{C_t, w_t\}} \frac{C_t^\gamma}{\gamma} && (15) \\
& + E_t \left[b_0 \delta \overline{C}_t^{\gamma - 1} w_t \widehat{v}(R_{t+1}, z_t) + \delta J(W_{t+1}, z_{t+1}) \right]
\end{aligned}
$$

First Order Conditions

- Differentiating with respect to C_t and w_t:

$$0 = C_t^{\gamma-1} - \delta R_f E_t \left[J_W \left(W_{t+1}, z_{t+1} \right) \right] \tag{16}$$

$$\begin{aligned} 0 &= E_t \left[b_0 \overline{C}_t^{\gamma-1} \widehat{v} \left(R_{t+1}, z_t \right) + J_W \left(W_{t+1}, z_{t+1} \right) \left(R_{t+1} - R_f \right) \right] \\ &= b_0 \overline{C}_t^{\gamma-1} E_t \left[\widehat{v} \left(R_{t+1}, z_t \right) \right] + E_t \left[J_W \left(W_{t+1}, z_{t+1} \right) R_{t+1} \right] \\ &\quad - R_f E_t \left[J_W \left(W_{t+1}, z_{t+1} \right) \right] \end{aligned} \tag{17}$$

Solution (continued)

- It is straightforward to show that (16) and (17) imply the standard envelope condition

$$C_t^{\gamma-1} = J_W\left(W_t, z_t\right) \tag{18}$$

- Substituting this into (16), one obtains the Euler equation

$$1 = \delta R_f E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\gamma-1}\right] \tag{19}$$

- Using (1) to compute the expectation in (19), we can solve for the risk-free interest rate:

$$R_f = e^{(1-\gamma)g_C - \frac{1}{2}(1-\gamma)^2\sigma_C^2}/\delta \tag{20}$$

Solution (continued)

- Using (18) and (19) in (17) implies

$$\begin{aligned} 0 &= b_0 \overline{C}_t^{\gamma-1} E_t\left[\widehat{v}\left(R_{t+1}, z_t\right)\right] + E_t\left[C_{t+1}^{\gamma-1} R_{t+1}\right] - R_f E_t\left[C_{t+1}^{\gamma-1}\right] \\ &= b_0 \overline{C}_t^{\gamma-1} E_t\left[\widehat{v}\left(R_{t+1}, z_t\right)\right] + E_t\left[C_{t+1}^{\gamma-1} R_{t+1}\right] - C_t^{\gamma-1}/\delta \quad (21) \end{aligned}$$

or

$$1 = b_0 \left(\frac{\overline{C}_t}{C_t}\right)^{\gamma-1} \delta E_t\left[\widehat{v}\left(R_{t+1}, z_t\right)\right] + \delta E_t\left[R_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{\gamma-1}\right] \quad (22)$$

- In equilibrium, (19) and (22) hold with individual consumption, C_t, replacing aggregate per-capita consumption, $\overline{C}_t$.

Solution (continued)

- Using (1) and (10), (22) is simplified to:

$$\begin{aligned} 1 &= b_0 \delta E_t \left[\widehat{v} \left(R_{t+1}, z_t \right) \right] \\ &\quad + \delta E_t \left[\frac{1 + f\left(z_{t+1}\right)}{f\left(z_t\right)} e^{g_D + \sigma_D \varepsilon_{t+1}} \left(e^{g_C + \sigma_C \eta_{t+1}} \right)^{\gamma - 1} \right] \end{aligned} \tag{23}$$

or

$$\begin{aligned} 1 &= b_0 \delta E_t \left[\widehat{v} \left(\frac{1 + f\left(z_{t+1}\right)}{f\left(z_t\right)} e^{g_D + \sigma_D \varepsilon_{t+1}}, z_t \right) \right] \\ &\quad + \delta e^{g_D - (1-\gamma) g_C + \frac{1}{2}(1-\gamma)^2 \sigma_C^2 \left(1 - \rho^2\right)} \\ &\quad \times E_t \left[\frac{1 + f\left(z_{t+1}\right)}{f\left(z_t\right)} e^{\left(\sigma_D - (1-\gamma)\rho\sigma_C\right)\varepsilon_{t+1}} \right] \end{aligned} \tag{24}$$

Solution (continued)

- The price-dividend ratio, $P_t/D_t = f_t(z_t)$, can be computed numerically from (24).
- However, because $z_{t+1} = 1 + \eta\left(z_t \frac{\overline{R}}{R_{t+1}} - 1\right)$ and $R_{t+1} = \frac{1+f(z_{t+1})}{f(z_t)} e^{g_D + \sigma_D \varepsilon_{t+1}}$, z_{t+1} depends upon z_t, $f(z_t)$, $f(z_{t+1})$, and ε_{t+1}:

$$z_{t+1} = 1 + \eta\left(z_t \frac{\overline{R} f(z_t) e^{-g_D - \sigma_D \varepsilon_{t+1}}}{1 + f(z_{t+1})} - 1\right) \qquad (25)$$

- Therefore, (24) and (25) need to be solved jointly and can be done by an iterative numerical technique for finding the function $f(\cdot)$.

Numerical Soution for Price/Dividend Ratio

- Start by guessing an initial function, $f^{(0)}$, and use it to solve for z_{t+1} in (25) for given z_t and ε_{t+1}.
- Then, a new candidate solution, $f^{(1)}$, is obtained using the following recursion that is based on (24):

$$
\begin{aligned}
f^{(i+1)}(z_t) &= \delta e^{g_D-(1-\gamma)g_C+\frac{1}{2}(1-\gamma)^2\sigma_C^2\left(1-\rho^2\right)} \times \\
& E_t\left[\left[1+f^{(i)}(z_{t+1})\right]e^{(\sigma_D-(1-\gamma)\rho\sigma_C)\varepsilon_{t+1}}\right] \\
& +f^{(i)}(z_t)\, b_0 \delta E_t\left[\widehat{v}\left(\frac{1+f^{(i)}(z_{t+1})}{f^{(i)}(z_t)}e^{g_D+\sigma_D\varepsilon_{t+1}}, z_t\right)\right]
\end{aligned}
$$

where the expectations are computed using a Monte Carlo simulation of the ε_{t+1}.

- Given $f^{(1)}$, z_{t+1} is solved again from (25) and the procedure is repeated until $f^{(i)}$ converges.

Model Results

- For reasonable parameter, $P_t/D_t = f_t\left(z_t\right)$ decreases in z_t: if there are prior risky asset gains (z_t is low), then investors are less risk averse and bid up the risky asset price.
- Using the estimated $f\left(\cdot\right)$, the unconditional distribution of stock returns is simulated by randomly generating ε_t's.
- This shows that since dividends and consumption follow separate processes, and stock prices have volatility exceeding that of dividends (fundamentals), stock volatility can be made substantially higher than consumption volatility.

Model Results (continued)

- Moreover, the effect of loss aversion generates a significant equity risk premium for reasonable values of γ.
- Because investors care about stock volatility, per se, a large equity premium can exist despite low stock-consumption correlation.
- Consistent with empirical research finding negative correlations in stock returns at long horizons, the model generates predictability in stock returns: returns tend to be higher following crashes (when z_t is high) and smaller following expansions (when z_t is low).

The Impact of Irrational Traders on Asset Prices

- The Kogan, Ross, Wang, and Westerfield (2006) model assumes some investors are fully rational but others are irrational because they suffer from systematic optimism or pessimism.
- The model shows that irrational investors may not necessarily lose wealth to rational investors and be driven out of the asset market.
- Even when irrational investors do not survive in the long run, their trading can significantly impact equilibrium asset prices for substantial periods.

Kogan, Ross, Wang, Westerfield Model Assumptions

- A simple endowment economy has two types of representative agents: rational agents and agents that are irrationally optimistic or pessimistic regarding risky-asset returns. Both maximize utility of consumption at a single, future date.
- **Technology**: The risky asset is a claim on a single, risky date $T > 0$ dividend payment, D_T. D_T is the date T realization of

$$dD_t / D_t = \mu dt + \sigma dz \tag{27}$$

  where μ and σ are constants, $\sigma > 0$, and $D_0 = 1$.
- Aggregate consumption at date T is $C_T = D_T$.
- All agents can buy or sell (issue) a zero-coupon bond in zero net supply that makes a default-free payment of 1 at date T.

Assumptions (continued)

- **Preferences and Beliefs**: Rational and irrational agents each have date 0 endowment equal to one-half of the risky asset and have constant relative risk aversion. For example, the rational agents maximize

$$E_0 \left[\frac{C_{r,T}^{\gamma}}{\gamma} \right] \tag{28}$$

where $\gamma < 1$ and $C_{r,T}$ is rational traders' date T consumption.
- While rational agents believe (27), irrational agents perceive

$$dD_t / D_t = \left(\mu + \sigma^2 \eta \right) dt + \sigma d\widehat{z} \tag{29}$$

where they believe $d\widehat{z}$ is a Brownian motion, whereas in reality, $d\widehat{z} = dz - \sigma \eta dt$. Note if the constant η is positive (*negative*), irrational traders are optimistic (*pessimistic*).

Irrational Agent Beliefs

- Hence, rather than the probability measure P that is generated by dz, irrational traders believe that the probability measure is generated by $d\widehat{z}$, which we refer to as the probability measure $\widehat{P}$.

- Therefore, an irrational individual's expected utility is

$$\widehat{E}_0 \left[\frac{C_{n,T}^{\gamma}}{\gamma} \right] \tag{30}$$

where $C_{n,T}$ is the date T consumption of the irrational trader.

Solution Technique

- The irrational agent's utility can be reinterpreted as the state-dependent utility of a rational individual.
- Girsanov's theorem implies $d\widehat{P}_T = (\xi_T/\xi_0)\, dP_T$ where if $\xi_0 = 1$, then

$$\begin{aligned} \xi_T &= \exp\left[\int_0^T \sigma\eta dz - \frac{1}{2}\int_0^T (\sigma\eta)^2\, ds\right] \\ &= e^{-\frac{1}{2}\sigma^2\eta^2 T + \sigma\eta(z_T - z_0)} \end{aligned} \quad (31)$$

- Since σ and η are constants, ξ_t is lognormal $d\xi/\xi = \sigma\eta dz$.

Utility of Irrational Agents

- Thus, the irrational agents's expected utility can be written as

$$\widehat{E}_0\left[\frac{C_{n,T}^{\gamma}}{\gamma}\right] = E_0\left[\xi_T \frac{C_{n,T}^{\gamma}}{\gamma}\right] \tag{32}$$

$$= E_0\left[e^{-\frac{1}{2}\sigma^2\eta^2 T + \sigma\eta(z_T - z_0)} \frac{C_{n,T}^{\gamma}}{\gamma}\right]$$

- (32) shows that the objective function of the irrational trader is observationally equivalent to that of a rational trader whose utility depends on z_T, which is the same state (Brownian motion uncertainty) determining the risky asset's dividend.

Martingale Approach to Consumption and Portfolio Choice

- Given market completeness, the martingale approach where lifetime utility contains only a terminal bequest can be applied. The two types of agents' first order conditions are

$$C_{r,T}^{\gamma-1} = \lambda_r M_T \tag{33}$$

$$\xi_T C_{n,T}^{\gamma-1} = \lambda_n M_T \tag{34}$$

  where λ_r and λ_n are the Lagrange multipliers for the rational and irrational agents, respectively.

- Substituting out for M_T, we can write

$$C_{r,T} = (\lambda \xi_T)^{-\frac{1}{1-\gamma}} C_{n,T} \tag{35}$$

  where we define $\lambda \equiv \lambda_r / \lambda_n$.

Market Equilibrium

- Market clearing at the terminal date implies

$$C_{r,T} + C_{n,T} = D_T \tag{36}$$

- Equations (35) and (36) allow us to write:

$$C_{r,T} = \frac{1}{1 + (\lambda \xi_T)^{\frac{1}{1-\gamma}}} D_T \tag{37}$$

- Substituting (37) into (35), we also obtain

$$C_{n,T} = \frac{(\lambda \xi_T)^{\frac{1}{1-\gamma}}}{1 + (\lambda \xi_T)^{\frac{1}{1-\gamma}}} D_T \tag{38}$$

Solution

- The parameter $\lambda = \lambda_r / \lambda_n$ is determined by the individuals' initial endowments of wealth, equal to $E_0 \left[C_{i,T} M_T / M_0 \right]$, $i = r, n$.
- Note that the date t price of the zero coupon bond that pays 1 at date $T > t$ is

$$P(t, T) = E_t \left[M_T / M_t \right] \tag{39}$$

- For analytical convenience, consider deflating all assets prices, including the individuals' initial wealths, by this zero-coupon bond price.
- Define $W_{r,0}$ and $W_{n,0}$ as the initial wealths, deflated by this zero-coupon bond price, of the rational and irrational individuals, respectively.

Deflated Initial Wealths

- The deflated wealth of the rational agent is

$$
\begin{aligned}
W_{r,0} &= \frac{E_0\left[C_{r,T} M_T / M_0\right]}{E_0\left[M_T / M_0\right]} = \frac{E_0\left[C_{r,T} M_T\right]}{E_0\left[M_T\right]} \\
&= \frac{E_0\left[C_{r,T} C_{r,T}^{\gamma-1} / \lambda_r\right]}{E_0\left[C_{r,T}^{\gamma-1} / \lambda_r\right]} = \frac{E_0\left[C_{r,T}^{\gamma}\right]}{E_0\left[C_{r,T}^{\gamma-1}\right]} \\
&= \frac{E_0\left[\left[1+(\lambda \xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma} D_T^{\gamma}\right]}{E_0\left[\left[1+(\lambda \xi_T)^{\frac{1}{1-\gamma}}\right]^{1-\gamma} D_T^{\gamma-1}\right]}
\end{aligned}
\tag{40}
$$

where in the second line of (40), (33) is used to substitute for M_T, and in the third line (37) is used to substitute for $C_{r,T}$.

Solution for Lagrange Multiplier

- A similar derivation that uses (34) and (38) leads to

$$W_{n,0} = \frac{E_0 \left[(\lambda \xi_T)^{\frac{1}{1-\gamma}} \left[1 + (\lambda \xi_T)^{\frac{1}{1-\gamma}} \right]^{-\gamma} D_T^{\gamma} \right]}{E_0 \left[\left[1 + (\lambda \xi_T)^{\frac{1}{1-\gamma}} \right]^{1-\gamma} D_T^{\gamma-1} \right]} \tag{41}$$

- Since agents begin with equal $\frac{1}{2}$ shares of the endowment, $W_{r,0} = W_{n,0}$. Equating the right-hand sides of (40) and (41) and noting that ξ_T satisfies (31) and

$$D_T / D_t = e^{\left[\mu - \frac{1}{2}\sigma^2\right](T-t) + \sigma(z_T - z_t)} \tag{42}$$

is also lognormally distributed, it can be shown that

$$\lambda = e^{-\gamma \eta \sigma^2 T} \tag{43}$$

Price of the Risky Asset

- Given λ, M_T/M_t is a constant times $\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{1-\gamma} D_T^{\gamma-1}$, which allows us to solve for the equilibrium price of the risky asset.
- Define S_t as the date $t<T$ price of the risky asset deflated by the price of the zero-coupon bond, and define $\varepsilon_{T,t} \equiv \lambda\xi_T = \xi_t e^{-\gamma\eta\sigma^2 T-\frac{1}{2}\sigma^2\eta^2(T-t)+\sigma\eta(z_T-z_t)}$. Then

$$S_t = \frac{E_t\left[D_T M_T/M_t\right]}{E_t\left[M_T/M_t\right]} = \frac{E_t\left[\left(1+\varepsilon_{T,t}^{\frac{1}{1-\gamma}}\right)^{1-\gamma} D_T^{\gamma}\right]}{E_t\left[\left(1+\varepsilon_{T,t}^{\frac{1}{1-\gamma}}\right)^{1-\gamma} D_T^{\gamma-1}\right]} \tag{44}$$

Analysis of the Results

- Though the rational and irrational agents' portfolio policies do not have a closed form solution, it can be shown that agents' demand for the risky asset, ω, satisfies $|\omega| \leq 1 + |\eta|(2-\gamma)/(1-\gamma)$.
- For the case of all rational agents, $\eta = 0$, then $\varepsilon_{T,t} = \xi_t = 1$ and from (44) the deflated stock price, $S_{r,t}$, is

$$S_{r,t} = \frac{E_t\left[D_T^{\gamma}\right]}{E_t\left[D_T^{\gamma-1}\right]} = D_t e^{\left[\mu-\sigma^2\right](T-t)+\sigma^2\gamma(T-t)} \tag{45}$$

$$= e^{\left[\mu-(1-\gamma)\sigma^2\right]T+\left[(1-\gamma)-\frac{1}{2}\right]\sigma^2 t+\sigma(z_t-z_0)}$$

- Itô's lemma shows that (45) implies:

$$dS_{r,t}/S_{r,t} = (1-\gamma)\,\sigma^2 dt + \sigma dz \tag{46}$$

Results (continued)

- Similarly, when all agents are irrational, $S_{n,t}$ satisfies

$$S_{n,t} = e^{\left[\mu-(1-\gamma-\eta)\sigma^2\right]T+\left[(1-\gamma-\eta)-\frac{1}{2}\right]\sigma^2 t+\sigma(z_t-z_0)} = S_{r,t}e^{\eta\sigma^2(T-t)} \tag{47}$$

and its rate of return follows the process

$$dS_{n,t}/S_{n,t} = (1-\gamma-\eta)\,\sigma^2 dt + \sigma dz \tag{48}$$

- Note that the effect of η is similar to γ, so that if η is positive, the higher expected dividend growth acts like lower risk aversion. The greater demand raises the deflated stock price and lowers its equilibrium expected rate of return.

Results (continued)

- Note that (46) and (48) indicate that when there is only one type of agent, the volatility of the risky asset's deflated return equals σ.
- In contrast, when both types of agents are in the economy, applying Itô's lemma to (44) it can be shown that the risky asset's volatility, $\sigma_{S,t}$, satisfies

$$\sigma \leq \sigma_{S,t} \leq \sigma \left(1 + |\eta|\right) \tag{49}$$

- The conclusion is that a diversity of beliefs has the effect of raising the equilibrium volatility of the risky asset.

Risky Asset Price with Logarithmic Utiliy

- When utility is logarithmic so that $\gamma = 0$, (44) simplifies to

$$
\begin{aligned}
S_t &= \frac{1 + E_t\left[\xi_T\right]}{E_t\left[\left(1 + \xi_T\right) D_T^{-1}\right]} \qquad (50) \\
&= D_t e^{\left[\mu - \sigma^2\right](T-t)} \frac{1 + \xi_t}{1 + \xi_t e^{-\eta\sigma^2(T-t)}} \\
&= e^{\left[\mu - \frac{1}{2}\sigma^2\right]T - \frac{1}{2}\sigma^2(T-t) + \sigma(z_t - z_0)} \frac{1 + \xi_t}{1 + \xi_t e^{-\eta\sigma^2(T-t)}}
\end{aligned}
$$

Rational Agents' Share of Wealth

- Define

$$\alpha_t \equiv \frac{W_{r,t}}{W_{r,t} + W_{n,t}} = \frac{W_{r,t}}{S_t} \tag{51}$$

as the proportion of total wealth owned by the rational individuals. Using (40) and (44), when $\gamma = 0$ it equals

$$\alpha_t = \frac{E_t\left[\left(1+\varepsilon_{T,t}^{\frac{1}{1-\gamma}}\right)^{-\gamma} D_T^{\gamma}\right]}{E_t\left[\left(1+\varepsilon_{T,t}^{\frac{1}{1-\gamma}}\right)^{1-\gamma} D_T^{\gamma}\right]} = \frac{1}{1+E_t\left[\xi_T\right]} = \frac{1}{1+\xi_t} \tag{52}$$

Mean and Volatility of Risky Asset with Log Utility

- Viewing S_t as a function of D_t and ξ_t as in the second line of (50), Itô's lemma can be applied to derive

$$\sigma_{S,t} = \sigma + \eta\sigma \left[\frac{1}{1 + e^{-\eta\sigma^2(T-t)} \left(\alpha_t^{-1} - 1 \right)} - \alpha_t \right] \tag{53}$$

and

$$\mu_{S,t} = \sigma_{S,t}^2 - \eta\sigma \left(1 - \alpha_t \right) \sigma_{S,t} \tag{54}$$

where we have used $\alpha_t = 1/\left(1 + \xi_t\right)$ to substitute out for ξ_t.

- Note that when $\alpha_t = 1$ or 0, (53) and (54) are consistent with (46) and (48) for the case of $\gamma = 0$.

Friedman Conjecture

- The model is used to study how $C_{n,T}/C_{r,T}$ is distributed as T becomes large.
- Milton Friedman (1953) conjectured that irrational traders cannot survive in a competitive market: the *relative extinction* of an irrational agent would occur if

$$\lim_{T \to \infty} \frac{C_{n,T}}{C_{r,T}} = 0 \quad \text{a.s.} \tag{55}$$

  which means that for arbitrarily small δ the probability of $\left| \lim_{T \to \infty} \frac{C_{n,T}}{C_{r,T}} \right| > \delta$ equals zero.
- An agent is said to *survive relatively* in the long run if relative extinction does not occur.

Survival/Extinction under Log Utility

- For log utility, irrational agents always suffer relative extinction. To see this, rearrange (35):

$$\frac{C_{n,T}}{C_{r,T}} = \left(\lambda \xi_T\right)^{\frac{1}{1-\gamma}} \tag{56}$$

and for $\gamma = 0$, (43) implies that $\lambda = 1$. Hence,

$$\begin{aligned} \frac{C_{n,T}}{C_{r,T}} &= \xi_T \\ &= e^{-\frac{1}{2}\sigma^2\eta^2 T + \sigma\eta(z_T - z_0)} \end{aligned} \tag{57}$$

Stong Law of Large Numbers

- Based on the strong law of large numbers for Brownian motions, it can be shown that for any value of b

$$\lim_{T \to \infty} e^{aT + b(z_T - z_0)} = \begin{cases} 0 & a < 0 \\ \infty & a > 0 \end{cases} \tag{58}$$

  where convergence occurs almost surely.

- Since $-\frac{1}{2}\sigma^2\eta^2 < 0$ in (57), equation (55) is proved.

- The intuition for relative extinction is linked to the specialness of log utility. The logarithmic rational agent maximizes at each date t:

$$E_t\left[\ln C_{r,T}\right] = E_t\left[\ln W_{r,T}\right] \tag{59}$$

Growth-Optimum Portfolio

- (59) is equivalent to maximizing the expected continuously compounded return:

$$E_t\left[\frac{1}{T-t}\ln\left(W_{r,T}/W_{r,t}\right)\right]=\frac{1}{T-t}\left[E_t\left[\ln\left(W_{r,T}\right)\right]-\ln\left(W_{r,t}\right)\right] \tag{60}$$

  since $W_{r,t}$ is known at date t and $T-t>0$.
- Thus, this portfolio policy maximizes $E_t\left[d\ln W_{r,t}\right]$ and is referred to as the "growth-optimum portfolio."
- Note that the rational and irrational agents' wealths satisfy

$$dW_{r,t}/W_{r,t} = \mu_{r,t}dt+\sigma_{r,t}dz \tag{61}$$

$$dW_{n,t}/W_{n,t} = \mu_{n,t}dt+\sigma_{n,t}dz \tag{62}$$

  where, in general, $\mu_{r,t}$, $\mu_{n,t}$, $\sigma_{r,t}$, and $\sigma_{n,t}$, are time varying.

Growth in Relative Wealths

- Applying Itô's lemma, it is straightforward to show

$$d\ln\left(\frac{W_{n,t}}{W_{r,t}}\right) = \left[\left(\mu_{n,t} - \frac{1}{2}\sigma_{n,t}^2\right) - \left(\mu_{r,t} - \frac{1}{2}\sigma_{r,t}^2\right)\right] dt + \left(\sigma_{n,t} - \sigma_{r,t}\right) dz$$
$$= E_t\left[d\ln W_{n,t}\right] - E_t\left[d\ln W_{r,t}\right] + \left(\sigma_{n,t} - \sigma_{r,t}\right) dz \quad (63)$$

- Since the irrational agents choose a portfolio policy that deviates from the growth-optimum portfolio, we know $E_t\left[d\ln W_{n,t}\right] - E_t\left[d\ln W_{r,t}\right] < 0$, and thus $E_t\left[d\ln\left(W_{n,t}/W_{r,t}\right)\right] < 0$, making $d\ln\left(W_{n,t}/W_{r,t}\right)$ a process that is expected to steadily decline as $t \longrightarrow \infty$, verifying Friedman's conjecture.

Survival for General CRRA Utility

- The presence of irrational agents can impact asset prices for substantial periods of time prior to becoming "extinct."
- Moreover, if $\gamma < 0$ Friedman's conjecture may not always hold. Computing (56) for the general case of $\lambda = e^{-\gamma\eta\sigma^2 T}$:

$$\begin{aligned} \frac{C_{n,T}}{C_{r,T}} &= (\lambda\xi_T)^{\frac{1}{1-\gamma}} \\ &= e^{-\left[\gamma\eta+\frac{1}{2}\eta^2\right]\frac{\sigma^2}{1-\gamma}T+\frac{\sigma\eta}{1-\gamma}(z_T-z_0)} \end{aligned} \tag{64}$$

- The limiting behavior of $C_{n,T}/C_{r,T}$ depends on the sign of $\left[\gamma\eta + \frac{1}{2}\eta^2\right]$ or $\eta\left(\gamma + \frac{1}{2}\eta\right)$.

Survival/Extinction for General CRRA Utility

- If $\gamma < 0$, the strong law of large numbers implies

$$\lim_{T \longrightarrow \infty} \frac{C_{n,T}}{C_{r,T}} = \begin{cases} 0 & \eta < 0 & \text{rational trader survives} \\ \infty & 0 < \eta < -2\gamma & \text{irrational trader survives} \\ 0 & -2\gamma < \eta & \text{rational trader survives} \end{cases} \tag{65}$$

- If the irrational agent is pessimistic ($\eta < 0$) or strongly optimistic ($\eta > -2\gamma$), he becomes relatively extinct.
- However, when the irrational agent is moderately optimistic ($0 < \eta < -2\gamma$), it is the rational agent who becomes relatively extinct!

Intuition for General Result

- The intuition is that when $\gamma < 0$, rational agents' demand for the risky asset is less than that of a log utility agent, so that their wealths grow more slowly.

- When the irrational agent is moderately optimistic $(0 < \eta < -2\gamma)$, her portfolio demand is relatively closer to the growth-optimal portfolio.

Extensions

- If agents were assumed to gain utility from interim consumption, this would reduce the growth of their wealth and affect their relative survivability.
- Also, systematic differences between rational and irrational agents' risk aversions could influence the model's conclusions.
- In addition, one might expect that irrational agents might learn over time of their mistakes.
- Lastly, the model considers only one form of irrationality: systematic optimism or pessimism.

Summary

- This note considered two equilibrium models that incorporate psychological biases or irrationality.
- While considered "behavioral finance" models, they can be solved using standard techniques.
- Currently, there is little consensus among financial economists regarding the importance of incorporating aspects of behavioral finance into asset pricing theories.