Mixing Diffusion and Jump Processes

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Introduction

- Using a mixture of jump and diffusion processes can model asset prices that are subject to large, discontinuous changes, as might occur following the revelation of important information.

- Itô's lemma can be extended to derive the process of a variable that is a function of a mixed jump-diffusion process.

- When asset prices follow jump-diffusion processes, markets for contingent claims will, in general, be incomplete, requiring additional assumptions for valuation.

Modeling Jumps in Continuous Time

- Consider the continuous-time process:

$$dS/S = (\mu - \lambda k)\, dt + \sigma\, dz + \gamma\,(Y)\, dq \qquad (1)$$

where dz is a standard Wiener (Brownian motion) process and $q\,(t)$ is a Poisson counting process that increases by 1 if a Poisson-distributed event occurs.

- $dq\,(t)$ satisfies

$$dq = \left\{ \begin{array}{ll} 1 & \text{if a jump occurs} \\ 0 & \text{otherwise} \end{array} \right. \qquad (2)$$

Model Assumptions cont'd

- During each time interval, dt, the probability that $q(t)$ augments by 1 is $\lambda(t)\,dt$, where $\lambda(t)$ is the Poisson intensity.

- When a Poisson event occurs, say, at date $\widehat{t}$, S changes discontinuously, equal to $dS = \gamma(Y)S$ where γ is a function of $Y(\widehat{t})$ which is a random variable realized at date $\widehat{t}$.

- Thus, if a Poisson event occurs at date $\widehat{t}$, then $dS(\widehat{t}) = S(\widehat{t}^+) - S(\widehat{t}^-) = \gamma(Y)S(\widehat{t}^-)$, or

$$S(\widehat{t}^+) = [1 + \gamma(Y)]S(\widehat{t}^-) \tag{3}$$

Model Assumptions cont'd

- If $\gamma(Y) > 0$, there is an upward jump in S; whereas if $\gamma(Y) < 0$, there is a downward jump in S.

- Define $k \equiv E[\gamma(Y)]$ as the expected proportional jump when a Poisson event occurs, so that the expected change in S from $\gamma(Y)\, dq$ over the time interval dt is $\lambda k\, dt$.

- If μ denotes the instantaneous total expected rate of return (rate of change) on S, then:

$$
\begin{aligned}
E[dS/S] &= E[(\mu - \lambda k)\, dt] + E[\sigma\, dz] + E[\gamma(Y)\, dq] \quad (4) \\
&= (\mu - \lambda k)\, dt + 0 + \lambda k\, dt = \mu\, dt
\end{aligned}
$$

Comment on Generality

- Jump-diffusion processes can be generalized to a multivariate setting where the process for $S(t)$ can depend on multiple Brownian motion and Poisson jump components.

- If $\lambda(t)$ depends on a random state variable $x(t)$, where for example, $dx(t)$ follows a diffusion process, then $\lambda(t, x(t))$ is called a doubly stochastic Poisson process or Cox process.

Itô's Lemma for Jump-Diffusions

- Let $c(S,\, t)$ be a twice-differentiable function of $S(t)$, where $S(t)$ follows (1). An extension of Itô's lemma implies:

$$
\begin{aligned}
dc \;=\;& c_s\left[(\mu - \lambda k)S\, dt \,+\, \sigma S\, dz\,\right] + \frac{1}{2}c_{ss}\sigma^2 S^2\, dt \,+\, c_t\, dt \\
& + \;\{c\left([1 + \gamma(Y)]\,S,\, t\right) - c(S,\, t)\}\, dq \qquad (5)
\end{aligned}
$$

where subscripts on c denote its partial derivatives.

- When S jumps, $c(S,\, t)$ jumps to $c\left([1 + \gamma(Y)]\,S,\, t\right)$.

- Denote $\mu_c dt$ as the instantaneous expected rate of return on c per unit time, that is, $E[dc/c] = \mu_c dt$. Also, define σ_c as the standard deviation of the instantaneous rate of return on c, conditional on a jump not occurring.

Itô's Lemma for Jump-Diffusions

- Then (5) is

$$dc/c = [\mu_c - \lambda k_c(t)] dt + \sigma_c dz + \gamma_c(Y) dq \qquad (6)$$

where

$$\mu_c \equiv \frac{1}{c}\left[c_s(\mu - \lambda k)S + \frac{1}{2}c_{ss}\sigma^2 S^2 + c_t\right] + \lambda k_c(t) \quad (7)$$

$$\sigma_c \equiv \frac{c_s}{c}\sigma S \qquad (8)$$

$$\gamma_c = [c([1 + \gamma(Y)]S, t) - c(S, t)]/c(S, t) \qquad (9)$$

$$k_c(t) \equiv E_t[c([1 + \gamma(Y)]S, t) - c(S, t)]/c(S, t) \quad (10)$$

Here, $k_c(t)$ is the expected proportional jump of the variable $c(S, t)$ given that a Poisson event occurs.

Valuing Contingent Claims

- For simplicity, assume that λ is constant over time and that $\gamma(Y) = (Y-1)$: at the time of a jump $S\left(\widehat{t}^-\right)$ goes to $S\left(\widehat{t}^+\right) = YS\left(\widehat{t}^-\right)$. Also assume that successive random jump sizes, $(\widetilde{Y}-1)$, are independently and identically distributed.

- If μ and σ are constants, so that the continuous component of $S(t)$ is lognormal, then conditional upon n jumps in the interval $(0, t)$:

$$\tilde{S}(t) = S(0)\, e^{\left(\mu - \frac{1}{2}\sigma^2 - \lambda k\right)t + \sigma(\tilde{z}_t - z_0)}\, \tilde{y}(n) \qquad (11)$$

where $\tilde{z}_t - z_0 \sim N(0, t)$. $\tilde{y}(0) = 1$ and $\tilde{y}(n) = \prod_{i=1}^{n} \tilde{Y}_i$ for $n \geq 1$ where $\{\tilde{Y}_i\}_{i=1}^{n}$ is a set of independent identically distributed jumps.

Hedge Portfolio

- Similar to a Black-Scholes hedge, consider a portfolio invested in the underlying asset, contingent claim, and risk-free asset having portfolio proportions ω_1, ω_2, and $\omega_3 = 1 - \omega_1 - \omega_2$, respectively.

- The portfolio's instantaneous rate of return is

$$
\begin{aligned}
dH/H &= \omega_1 \, dS/S + \omega_2 \, dc/c + (1 - \omega_1 - \omega_2) r \, dt \qquad (12) \\
&= \left[\omega_1(\mu - r) + \omega_2(\mu_c - r) + r - \lambda(\omega_1 k + \omega_2 k_c) \right] dt \\
&\quad + (\omega_1 \sigma + \omega_2 \sigma_c) \, dz + \left[\omega_1 \gamma(Y) + \omega_2 \gamma_c(Y) \right] dq
\end{aligned}
$$

Imperfect Hedge

- Though jumps in the asset and the contingent claim occur simultaneously, their relative size, $\gamma\left(\widetilde{Y}\right)/\gamma_c\left(\widetilde{Y}\right)$, is unpredictable due to possible nonlinearities.

- Hence, a predetermined hedge ratio, ω_1/ω_2, that would eliminate all portfolio risk does not exist.

- The implication is that one cannot perfectly replicate the contingent claim's payoff by a portfolio composed of the underlying asset and the risk-free asset, making the market incomplete.

Imperfect Hedge cont'd

- Suppose one sets $\omega_1^*/\omega_2^* = -\sigma_c/\sigma = -c_s S/c$ to eliminate only the Brownian motion risk. This leads to:

$$
\begin{aligned}
dH/H &= [\omega_1^*(\mu - r) + \omega_2^*(\mu_c - r) + r - \lambda(\omega_1^* k + \omega_2^* k_c)]\, dt \\
&\quad + [\omega_1^* \gamma(Y) + \omega_2^* \gamma_c(Y)]\, dq
\end{aligned} \tag{13}
$$

- Using the definitions of γ, γ_c, and $\omega_1^* = -\omega_2^* c_s S/c$, the jump term, $[\omega_1^* \gamma(Y) + \omega_2^* \gamma_c(Y)]\, dq$, then equals

$$
\begin{cases}
\omega_2^* \left[\dfrac{c(S\tilde{Y}, t) - c(S, t)}{c(S, t)} - c_s(S, t)\dfrac{S\tilde{Y} - S}{c(S, t)} \right] & \text{if a jump occurs} \\
0 & \text{otherwise}
\end{cases}
$$

$$\tag{14}$$

An Imperfect Hedge cont'd

- Consider the pattern of profits and losses on the (quasi-) hedge portfolio if the contingent claim is a European option on a stock with a time until maturity of τ and a strike price X.

- If as in (1), the rate of return on the stock is independent of its price level, then the absence of arbitrage restricts the option price to a convex function of the asset price.

- As shown in Figure 11.1, the option's convexity implies that $c(SY, t) - c(S, t) - c_s(S, t)[SY - S] \geq 0$ for all Y and t.

Jump Risk
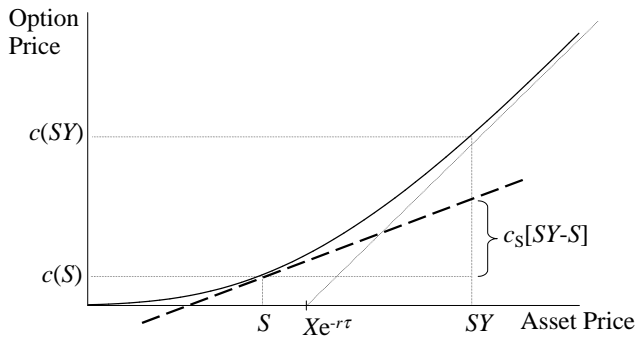
An Imperfect Hedge cont'd

- This fact and (14) implies that the unanticipated return on the hedge portfolio has the same sign as ω_2^*, so that the expected portfolio jump size, $\omega_1^* k + \omega_2^* k_c$, also has the same sign as ω_2^*.

- Thus, an option writer who implements the hedge earns, most of the time, more than the portfolio's expected rate of return.

- However, on those rare occasions when the underlying asset price jumps, a relatively large loss is incurred.

Diversifiable Jump Risk

- The hedge portfolio is exposed to jump risk which, in general, may have a "market price."

- However, if one assumes that this jump risk is the result of purely firm specific information and, hence, is perfectly diversifiable, it would have a market price of zero.

- In this case, the hedge portfolio's expected rate of return must equal the risk-free rate, r:

$$\omega_1^*(\mu - r) + \omega_2^*(\mu_c - r) + r = r \tag{15}$$

or

$$\omega_1^*/\omega_2^* = -\sigma_c/\sigma = -(\mu_c - r)/(\mu - r) \tag{16}$$

Diversifiable Jump Risk

- Let the contingent claim's time until maturity be $\tau \equiv T - t$, and write its price as $c(S, \tau)$.

- Using (16) and substituting in for μ_c and σ_c from the definitions (7) and (8), we obtain:

$$\frac{1}{2}\sigma^2 S^2 c_{ss} + (r - \lambda k)S c_s - c_\tau - rc + \lambda E_t \left[c(S\tilde{Y}, \tau) - c(S, \tau) \right] = 0 \tag{17}$$

Diversifiable Jump Risk

- For a call option, this is solved subject to the boundary conditions $c(0, \tau) = 0$ and $c(S(T), 0) = \max[S(T) - X, 0]$.

- Note that when $\lambda = 0$, equation (17) is the standard Black-Scholes equation, which has the solution

$$b(S, \tau, X, \sigma^2, r) \equiv S\, N(d_1) - Xe^{-r\tau}\, N(d_2) \qquad (18)$$

where $d_1 = [\ln(S/X) + (r + \frac{1}{2}\sigma^2)\tau] / (\sigma\sqrt{\tau})$ and $d_2 = d_1 - \sigma\sqrt{\tau}$.

Diversifiable Jump Risk

- Robert Merton (1976) shows that the general solution to (17) is

$$c(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda \tau}(\lambda \tau)^n}{n!} E_t \left[b(S \, \tilde{y}(n) e^{-\lambda k \tau}, \tau, X, \sigma^2, r) \right]$$

(19)

where recall, $\tilde{y}(0) = 1$ and $\tilde{y}(n) = \prod_{i=1}^{n} \tilde{Y}_i$ for $n \geq 1$.

- The intuition behind the formula in (19) is that the option is a probability-weighted average of expected Black-Scholes option prices.

Lognormal Jump Proportions

- Suppose $\tilde{Y}$ is lognormally distributed, where
  $E[\ln \tilde{Y}] \equiv \alpha - \frac{1}{2}\delta^2$ where $var[\ln \tilde{Y}] \equiv \delta^2$, so that
  $E[\tilde{Y}] = e^\alpha = 1 + k$.

- Hence, $\alpha \equiv \ln(1 + k)$. Also if μ is assumed to be constant,
  the probability density for $\ln[S(t + \tau)]$, conditional on the
  value of $S(t)$, is

$$\sum_{n=0}^{\infty} g(\ln[S(t + \tau)/S(t)] \,|\, n)h(n) \qquad (20)$$

where $g(\cdot \,|\, n)$ is the conditional density function given that n
jumps occur during the interval between t and $t + \tau$, and $h(n)$
is the probability that n jumps occur between t and $t + \tau$.

Lognormal Jump Proportions

- The values of these expressions are

$$g\left(\ln\left[\frac{S(t+\tau)}{S(t)}\right] \mid n\right) \equiv \frac{\exp\left[-\frac{\left(\ln\left[\frac{S(t+\tau)}{S(t)}\right] - \left(\mu - \lambda k + \frac{n\alpha}{\tau} - \frac{\nu_n^2}{2}\right)\tau\right)^2}{2\nu_n^2\tau}\right]}{\sqrt{2\pi\nu_n^2\tau}} \quad (21)$$

$$h(n) \equiv \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \quad (22)$$

where $\nu_n^2 \equiv \sigma^2 + n\delta^2/\tau$ is the "average" variance per unit time.

Solution

- Setting $\mu = r$ allows us to compute the date t risk-neutral expectation of $\max[S(T) - X, 0]$, discounted by the risk-free rate, and conditional on n jumps occurring:

$$E_t[\, b(S\tilde{y}(n)e^{-\lambda k \tau}, \tau, X, \sigma^2, r)] = e^{-\lambda k \tau}(1 + k)^n \, b_n(S, \tau) \tag{23}$$

where $b_n(S, \tau) \equiv b(S, \tau, X, \nu_n^2, r_n)$ and where $r_n \equiv r - \lambda k + n\alpha/\tau$.

Solution

- The value of the option is then the weighted average of these conditional values, where each weight equals the probability that a Poisson random variable with characteristic parameter $\lambda\tau$ will equal n.

- Defining $\lambda' \equiv \lambda(1 + k)$, this equals

$$
\begin{aligned}
c(S,\,\tau) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\, e^{-\lambda k\tau}\,(1+k)^n\, b_n(S,\,\tau) \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!}\, b_n(S,\,\tau) \qquad (24)
\end{aligned}
$$

Nondiversifiable Jump Risk

- In some cases, such as a stock market crash, it is unrealistic to assume a zero market price of jump risk.

- Equilibrium models, such as Bates (1991) and Naik and Lee (1990), can be used to derive the equilibrium market price of jump risk when aggregate wealth is subject to jumps.

- Under particular assumptions, formulas for options will take a form similar to (24).

Black-Scholes versus Jump-Diffusion Model

- Empirically, the standard Black-Scholes model underprices out-of-the-money and in-the-money options relative to at-the-money-options, a phenomenon referred to as a *volatility smile* or *volatility smirk*.

- A model that permits the underlying asset's price to jump (positive or negative) can generate an asset price distribution that has fatter tails than the lognormal.

- With extreme price changes more likely, a jump-diffusion option pricing model can better match the market prices of many types of options.

Black-Scholes versus Jump-Diffusion Model

- To account for time variation in implied option volatility, *stochastic volatility* option pricing models assume that the underlying asset price follows a diffusion process such as $dS/S = \mu dt + \sigma_t dz$.
- However, the volatility, σ_t, follows a mean-reverting process of the form $d\sigma_t = \alpha(\sigma_t) dt + \beta(\sigma_t) dz_\sigma$, where dz_σ is another Brownian motion process possibly correlated with dz.
- Similar to the jump-diffusion model, one must assign a market price of risk associated with the volatility uncertainty reflected in the dz_σ term.
- To capture both time variation in volatilities and volatility smiles and smirks, it appears that an option price model permitting both stochastic volatility and jumps is required (Bates (2002), Bakshi, Cao, and Chen (1997)).

Summary

- The mixed jump-diffusion process captures more realistic asset price dynamics, but the market for the asset and its contingent claim will, in general, be incomplete.
- Additional theory is needed to assign a market price of jump risk.
- However, since the actual prices of many types of options appear to reflect the likelihood of extreme movements in the underlying asset's price, the jump-diffusion model has better empirical performance.
- We next study continuous-time consumption and portfolio choices, which will allow us to derive assets' equilibrium risk premia in a continuous-time economy.