Arbitrage, Martingales, and Pricing Kernels

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Introduction

- A contingent claim's price process can be transformed into a *martingale* process by
  1. Adjusting its Brownian motion by the market price of risk.
  2. Deflating by a riskless asset price.
- The claim's value equals the expectation of the transformed process's future payoff.
- We derive the continuous-time state price deflator that transforms actual probabilities into risk-neutral probabilities.
- Valuing a contingent claim might be simplified by deflating the contingent claim's price by that of another risky asset.
- We consider applications: options on assets that pay a continuous dividend; the term structure of interest rates.

Arbitrage and Martingales

- Let S be the value of a risky asset that follows a general scalar diffusion process

$$dS = \mu S dt + \sigma S dz \tag{1}$$

where both $\mu = \mu(S,t)$ and $\sigma = \sigma(S,t)$ may be functions of S and t and dz is a Brownian motion.

- Itô's lemma gives the process for a contingent claim's price, $c(S,t)$:

$$dc = \mu_c c dt + \sigma_c c dz \tag{2}$$

where $\mu_c c = c_t + \mu S c_S + \frac{1}{2}\sigma^2 S^2 c_{SS}$ and $\sigma_c c = \sigma S c_S$, and the subscripts on c denote partial derivatives.

- Consider a hedge portfolio of -1 units of the contingent claim and c_S units of the risky asset.

Arbitrage and Martingales cont'd

- The value of this hedge portfolio, H, satisfies

$$H = -c + c_S S \tag{3}$$

and the change in its value over the next instant is

$$\begin{aligned} dH &= -dc + c_S dS \\ &= -\mu_c c dt - \sigma_c c dz + c_S \mu S dt + c_S \sigma S dz \\ &= [c_S \mu S - \mu_c c]\, dt \end{aligned} \tag{4}$$

- In the absence of arbitrage, the riskless portfolio change must be $H(t)r(t)dt$:

$$dH = [c_S \mu S - \mu_c c]\, dt = rHdt = r[-c + c_S S]dt \tag{5}$$

Arbitrage and Martingales cont'd

- This no-arbitrage condition for dH implies:

$$c_S \mu S - \mu_c c = r[-c + c_S S] \tag{6}$$

- Substituting $\mu_c c = c_t + \mu S c_S + \frac{1}{2}\sigma^2 S^2 c_{SS}$ into (6) leads to the Black-Scholes equation:

$$\frac{1}{2}\sigma^2 S^2 c_{SS} + rSc_S - rc + c_t = 0 \tag{7}$$

- However, a different interpretation of (6) results from substituting $c_S = \frac{\sigma_c c}{\sigma S}$ (from $\sigma_c c = \sigma S c_S$):

$$\frac{\mu - r}{\sigma} = \frac{\mu_c - r}{\sigma_c} \equiv \theta(t) \tag{8}$$

- No-arbitrage condition (8) requires a unique market price of risk, say $\theta(t)$, so that $\mu_c = r + \sigma_c \theta(t)$.

A Change in Probability

- Substituting for μ_c in (2) gives

$$dc = \mu_c c dt + \sigma_c c dz = [rc + \theta\sigma_c c]\, dt + \sigma_c c dz \tag{9}$$

- Next, consider a new process $\widehat{z}_t = z_t + \int_0^t \theta(s)\, ds$, so that $d\widehat{z}_t = dz_t + \theta(t)\, dt$.
- Then substituting $dz_t = d\widehat{z}_t - \theta(t)\, dt$ in (9):

$$\begin{aligned} dc &= [rc + \theta\sigma_c c]\, dt + \sigma_c c\, [d\widehat{z} - \theta dt] \\ &= rc dt + \sigma_c c d\widehat{z} \end{aligned} \tag{10}$$

- If $\widehat{z}_t$ were a Brownian motion, future values of c generated by $d\widehat{z}$ occur under the Q or "risk-neutral" probability measure.
- The actual or "physical" distribution, P, is generated by the dz Brownian motion.

Girsanov's Theorem

- Let dP_T be the instantaneous change in the cumulative distribution at date T generated by dz_t (the physical pdf).
- dQ_T is the analogous risk-neutral pdf generated by $d\widehat{z}_t$.
- Girsanov's theorem says that at date $t < T$, the two probability densities satisfy

$$\begin{aligned} dQ_T &= \exp\left[-\int_t^T \theta(u)\, dz - \frac{1}{2}\int_t^T \theta(u)^2\, du\right] dP_T \\ &= (\xi_T/\xi_t)\, dP_T \end{aligned} \tag{11}$$

where ξ_t is a positive random process depending on $\theta(t)$ and z_t:

$$\xi_\tau = \exp\left[-\int_0^\tau \theta(u)\, dz - \frac{1}{2}\int_0^\tau \theta(u)^2\, du\right] \tag{12}$$

Girsanov's Theorem cont'd

- Thus, multiplying the physical pdf, dP_T, by ξ_T/ξ_t leads to the risk-neutral pdf, dQ_T, and since $\xi_T/\xi_t > 0$, whenever dP_T has positive probability, so does dQ_T, making them *equivalent* measures.
- Yet if $\theta(t) > 0$, then a positive dz innovation lowers ξ_τ, making the risk-neutral probability of a high z_t state less than its physical probability.
- Rearranging (11) gives the Radon-Nikodym derivative:

$$\frac{dQ_T}{dP_T} = \xi_T/\xi_t \tag{13}$$

- Later we will relate this derivative to the continuous-time pricing kernel.

Money Market Deflator

- Let $B(t)$ be the value of an instantaneous-maturity riskless "money market fund" investment:

$$dB/B = r(t)dt \tag{14}$$

- Note that $B(T) = B(t)\, e^{\int_t^T r(u)du}$ for any date $T \geq t$.
- Now define $C(t) \equiv c(t)/B(t)$ as the deflated price process for the contingent claim and use Itô's lemma:

$$\begin{aligned} dC &= \frac{1}{B}dc - \frac{c}{B^2}dB \\ &= \frac{rc}{B}dt + \frac{\sigma_c c}{B}d\widehat{z} - r\frac{c}{B}dt \\ &= \sigma_c C d\widehat{z} \end{aligned} \tag{15}$$

since $dcdB = 0$ and we substitute for dc from (10).

Money Market Deflator cont'd

- An implication of (15) is

$$C(t) = \widehat{E}_t [C(T)] \quad \forall T \geq t \tag{16}$$

where $\widehat{E}_t [\cdot]$ denotes the expectation operator under the probability measure generated by $d\widehat{z}$.

- Thus, $C(t)$ is a martingale (random walk) process.
- Note that (16) holds for any deflated non-dividend-paying contingent claim, including $C = \frac{S}{B}$.
- Later, we will consider assets that pay dividends.

Feynman-Kac Solution

- Rewrite (16) in terms of the undeflated contingent claims price:

$$\begin{aligned} c(t) &= B(t)\widehat{E}_t\left[c\left(T\right)\frac{1}{B\left(T\right)}\right] \\ &= \widehat{E}_t\left[e^{-\int_t^T r(u)du}c\left(T\right)\right] \end{aligned} \qquad (17)$$

- Equation (17) is the "Feynman-Kac" solution to the Black-Scholes PDE and does not require knowledge of $\theta(t)$.
- This is the continuous-time formulation of risk-neutral pricing: risk-neutral (or Q measure) expected payoffs are discounted by the risk-free rate.

Arbitrage and Pricing Kernels

- Recall from the single- or multi-period consumption-portfolio choice problem with time-separable utility:

$$\begin{aligned} c(t) &= E_t\left[m_{t,T}\, c(T)\right] \\ &= E_t\left[\frac{M_T}{M_t} c(T)\right] \end{aligned} \tag{18}$$

where date $T \geq t$, $m_{t,T} \equiv M_T / M_t$ and $M_t = U_c(C_t, t)$.

- Rewriting (18):

$$c(t)\, M_t = E_t\left[c(T)\, M_T\right] \tag{19}$$

which says that the deflated price process, $c(t)\, M_t$, is a martingale under P (not Q).

Arbitrage and Pricing Kernels cont'd

- Assume that the state price deflator, M_t, follows a strictly positive diffusion process of the general form

$$dM_t = \mu_m dt + \sigma_m dz \tag{20}$$

- Define $c^m = cM$ and apply Itô's lemma:

$$\begin{aligned} dc^m &= cdM + Mdc + (dc)(dM) \qquad (21) \\ &= [c\mu_m + M\mu_c c + \sigma_c c\sigma_m]\, dt + [c\sigma_m + M\sigma_c c]\, dz \end{aligned}$$

- If $c^m = cM$ satisfies (19), that is, c^m is a martingale, then its drift in (21) must be zero, implying

$$\mu_c = -\frac{\mu_m}{M} - \frac{\sigma_c \sigma_m}{M} \tag{22}$$

Arbitrage and Pricing Kernels cont'd

- Consider the case in which c is the instantaneously riskless investment $B(t)$; that is, $dc(t) = dB(t) = r(t)Bdt$ so that $\sigma_c = 0$ and $\mu_c = r(t)$.
- From (22), this requires

$$r(t) = -\frac{\mu_m}{M} \tag{23}$$

- Thus, the expected rate of change of the pricing kernel must equal minus the instantaneous risk-free interest rate.
- Next, consider the general case where the asset c is risky, so that $\sigma_c \neq 0$. Using (22) and (23) together, we obtain

$$\mu_c = r(t) - \frac{\sigma_c \sigma_m}{M} \tag{24}$$

or

Arbitrage and Pricing Kernels cont'd

$$\frac{\mu_c - r}{\sigma_c} = -\frac{\sigma_m}{M} \tag{25}$$

- Comparing (25) to (8), we see that

$$-\frac{\sigma_m}{M} = \theta(t) \tag{26}$$

- Thus, the no-arbitrage condition implies that the form of the pricing kernel must be

$$dM/M = -r(t)\,dt - \theta(t)\,dz \tag{27}$$

Arbitrage and Pricing Kernels cont'd

- Define $m_t \equiv \ln M_t$ so that $dm = -[r + \frac{1}{2}\theta^2]dt - \theta dz$.
- We can rewrite (18) as

$$\begin{aligned} c(t) &= E_t[c(T) M_T / M_t] = E_t\left[c(T) e^{m_T - m_t}\right] \quad (28) \\ &= E_t\left[c(T) e^{-\int_t^T \left[r(u) + \frac{1}{2}\theta^2(u)\right]du - \int_t^T \theta(u) dz}\right] \end{aligned}$$

- Since the price under the money-market deflator (Q measure) and the SDF (P measure) must be the same, equating (17) and (28) implies

$$\widehat{E}_t\left[e^{-\int_t^T r(u)du} c(T)\right] = E_t[c(T) M_T / M_t] \quad (29)$$

$$= E_t\left[e^{-\int_t^T r(u)du} c(T) e^{-\int_t^T \frac{1}{2}\theta^2(u)du - \int_t^T \theta(u)dz}\right]$$

Linking Valuation Methods

- Substituting the definition of ξ_τ from (12) leads to

$$\begin{aligned} \widehat{E}_t\left[e^{-\int_t^T r(u)du}c(T)\right] &= E_t\left[e^{-\int_t^T r(u)du}c(T)(\xi_T/\xi_t)\right] \\ \widehat{E}_t[C(T)] &= E_t[C(T)(\xi_T/\xi_t)] \qquad (30) \\ \int C(T)\,dQ_T &= \int C(T)(\xi_T/\xi_t)\,dP_T \end{aligned}$$

where $C(t) = c(t)/B(t)$. Thus, relating (29) to (30):

$$M_T/M_t = e^{-\int_t^T r(u)du}(\xi_T/\xi_t) \qquad (31)$$

- Hence, M_T/M_t provides both discounting at the risk-free rate and transforming the probability distribution to the risk-neutral one via ξ_T/ξ_t.

Multivariate Case

- Consider a multivariate extension where asset returns depend on an $n \times 1$ vector of independent Brownian motion processes, $\mathbf{dZ} = (dz_1 ... dz_n)'$ where $dz_i dz_j = 0$ for $i \neq j$.
- A contingent claim whose payoff depended on these asset returns has the price process

$$dc/c = \mu_c dt + \Sigma_c \mathbf{dZ} \tag{32}$$

where Σ_c is a $1 \times n$ vector $\Sigma_c = (\sigma_{c1} ... \sigma_{cn})$.

- Let the corresponding $n \times 1$ vector of market prices of risks associated with each of the Brownian motions be $\Theta = (\theta_1 ... \theta_n)'$.

Multivariate Case cont'd

- Then the no-arbitrage condition (the multivariate equivalent of (8)) is

$$\begin{aligned} \mu_c - r &= \Sigma_c \Theta \\ &= \sum_{i=1}^{n} \sigma_{ci} \theta_i \end{aligned} \tag{33}$$

- Equations (16) and (17) would still hold, and now the pricing kernel's process would be given by

$$\begin{aligned} dM/M &= -r(t)\,dt - \Theta(t)'\,\mathbf{dZ} \\ &= -r(t)\,dt - \sum_{i=1}^{n} \theta_i dz_i \end{aligned} \tag{34}$$

Alternative Price Deflators

- Consider an option written on the difference between two securities' (stocks') prices. The date t price of stock 1, $S_1(t)$, follows the process

$$dS_1/S_1 = \mu_1 dt + \sigma_1 dz_1 \tag{35}$$

and the date t price of stock 2, $S_2(t)$, follows the process

$$dS_2/S_2 = \mu_2 dt + \sigma_2 dz_2 \tag{36}$$

where σ_1 and σ_2 are assumed to be constants and $dz_1 dz_2 = \rho dt$.

- Let $C(t)$ be the date t price of a European option written on the difference between these two stocks' prices.

Alternative Price Deflators cont'd

- At this option's maturity date, T, its value equals

$$C(T) = \max[0, S_1(T) - S_2(T)] \tag{37}$$

- Now define $c(t) = C(t)/S_2(t)$, $s(t) \equiv S_1(t)/S_2(t)$, and $B(t) = S_2(t)/S_2(t) = 1$ as the deflated price processes, where the prices of the option, stock 1, and stock 2 are all normalized by the price of stock 2.
- Under this normalized price system, the payoff (37) is

$$c(T) = \max[0, s(T) - 1] \tag{38}$$

- Applying Itô's lemma, the process for $s(t)$ is

$$ds/s = \mu_s dt + \sigma_s dz_3 \tag{39}$$

Alternative Price Deflators cont'd

- Here $\mu_s \equiv \mu_1 - \mu_2 + \sigma_2^2 - \rho\sigma_1\sigma_2$, $\sigma_s dz_3 \equiv \sigma_1 dz_1 - \sigma_2 dz_2$, and $\sigma_s^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$.
- Further, when prices are measured in terms of stock 2, the deflated price of stock 2 becomes the riskless asset with $dB/B = 0dt$ (the deflated price never changes).
- Using Itô's lemma on c,

$$dc = \left[c_s\, \mu_s s + c_t + \frac{1}{2} c_{ss}\, \sigma_s^2 s^2 \right] dt + c_s\, \sigma_s s\, dz_3 \tag{40}$$

- The familiar Black-Scholes hedge portfolio can be created from the option and stock 1. The portfolio's value is

$$H = -c + c_s s \tag{41}$$

Alternative Price Deflators cont'd

- The instantaneous change in value of the portfolio is

$$\begin{aligned} dH &= -dc + c_s ds \\ &= -\left[c_s \mu_s s + c_t + \frac{1}{2} c_{ss} \sigma_s^2 s^2 \right] dt - c_s \sigma_s s \, dz_3 \\ &\quad + c_s \mu_s s \, dt + c_s \sigma_s s \, dz_3 \\ &= -\left[c_t + \frac{1}{2} c_{ss} \sigma_s^2 s^2 \right] dt \end{aligned} \tag{42}$$

which is riskless and must earn the riskless return $dB/B = 0$:

$$dH = -\left[c_t + \frac{1}{2} c_{ss} \sigma_s^2 s^2 \right] dt = 0 \tag{43}$$

which implies

$$c_t + \frac{1}{2} c_{ss} \sigma_s^2 s^2 = 0 \tag{44}$$

Alternative Price Deflators cont'd

- This is the Black-Scholes PDE with the risk-free rate, r, set to zero. With boundary condition (38), the solution is

$$c(s,\ t)\ =\ s\,N(d_1)\ -\ N(d_2) \tag{45}$$

where

$$d_1 \quad = \quad \frac{\ln\left(s(t)\right)\ +\ \frac{1}{2}\sigma_s^2\left(T-t\right)}{\sigma_s\sqrt{T-t}} \tag{46}$$

$$d_2 \quad = \quad d_1\ -\ \sigma_s\sqrt{T-t}$$

- Multiply by $S_2\left(t\right)$ to convert back to the undeflated price system:

$$C(\,t)\ =\ S_1\,N(d_1)\ -\ S_2\,N(d_2) \tag{47}$$

- $C(t)$ does not depend on $r(t)$, so that this formula holds even for stochastic interest rates.

Continuous Dividends

- Let $S(t)$ be the date t price per share of an asset that continuously pays a dividend of $\delta S(t)$ per unit time. Thus,

$$dS = (\mu - \delta) Sdt + \sigma Sdz \tag{48}$$

  where σ and δ are assumed to be constants.
- Note that the asset's total rate of return is $dS/S + \delta dt = \mu dt + \sigma dz$, so that μ is its instantaneous expected rate of return.
- Consider a European call option written on this asset with exercise price of X and maturity date of $T > t$, where we define $\tau \equiv T - t$.
- Let r be the constant risk-free interest rate.

Continuous Dividends cont'd

- Based on (17), the date t price of this option is

$$\begin{aligned} c(t) &= \widehat{E}_t \left[e^{-r\tau} c(T) \right] \\ &= e^{-r\tau} \widehat{E}_t \left[\max \left[S(T) - X, 0 \right] \right] \end{aligned} \tag{49}$$

- In the absence of arbitrage, the Black-Scholes hedging argument requires equation (8) which implies

$$\mu = r + \sigma\theta(t). \tag{50}$$

- In addition, define $d\widehat{z} = dz + \theta(t)\, dt$ as the Q-measure Brownian motion.

Continuous Dividends cont'd

- Making these substitutions for μ and dz into equation (48) leads to

$$\begin{aligned} dS &= \left(r + \sigma\theta\left(t\right) - \delta\right) Sdt + \sigma S\left(d\widehat{z} - \theta\left(t\right) dt\right) \\ &= \left(r - \delta\right) Sdt + \sigma Sd\widehat{z} \end{aligned} \tag{51}$$

- Since $r - \delta$ and σ are constants, S is a geometric Brownian motion process and is lognormally distributed under Q.

Continuous Dividends cont'd

- Thus, the risk-neutral distribution of $\ln[S(T)]$ is normal:

$$\ln[S(T)] \sim N\left(\ln[S(t)] + (r - \delta - \frac{1}{2}\sigma^2)\tau,\ \sigma^2\tau\right) \quad (52)$$

- Equation (49) can now be computed as

$$\begin{aligned} c(t) &= e^{-r\tau}\widehat{E}_t\left[\max\left[S(T) - X, 0\right]\right] \quad (53) \\ &= e^{-r\tau}\int_X^{\infty}(S(T) - X)\, g(S(T))\, dS(T) \end{aligned}$$

where $g(S_T)$ is the lognormal probability density function.

- Consider the change in variable

$$Y = \frac{\ln[S(T)/S(t)] - \left(r - \delta - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \quad (54)$$

Continuous Dividends cont'd

- $Y \sim N(0,1)$ and allows (53) to be evaluated as

$$c = Se^{-\delta\tau}N(d_1) - Xe^{-r\tau}N(d_2) \tag{55}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/X) + \left(r - \delta + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ d_2 &= d_1 - \sigma\sqrt{\tau} \end{aligned} \tag{56}$$

- If contingent claims have more complex payoffs or the underlying asset has a more complex risk-neutral process, a numeric solution to $c(t) = \widehat{E}_t\left[e^{-r\tau}c(S(T))\right]$ can be obtained, perhaps by Monte Carlo simulation.

Continuous Dividends cont'd

- Compared to an option written on an asset that pays no dividends, the non-dividend-paying asset's price, $S(t)$, is replaced with the dividend-discounted price of the dividend-paying asset, $S(t)e^{-\delta\tau}$ (to keep the total expected rate of return at r).
- Thus, the risk-neutral expectation of $S(T)$ is

$$\begin{aligned} \hat{E}_t[S(T)] &= S(t)e^{(r-\delta)\tau} \\ &= S(t)e^{-\delta\tau}e^{r\tau} = \overline{S}(t)e^{r\tau} \end{aligned} \tag{57}$$

where we define $\overline{S}(t) \equiv S(t)e^{-\delta\tau}$.

Foreign Currency Options

- Define $S(t)$ as the domestic currency value of a unit of foreign currency (spot exchange rate).
- Purchase of a foreign currency allows the owner to invest at the risk-free foreign currency interest rate, r_f.
- Thus the dividend yield will equal this foreign currency rate, $\delta = r_f$ and $\hat{E}_t[S(T)] = S(t)\, e^{(r-r_f)\tau}$.
- This expression is the no-arbitrage value of the date t forward exchange rate having a time until maturity of τ, that is, $F_{t,\tau} = Se^{(r-r_f)\tau}$.
- Therefore, a European option on foreign exchange is

$$c(t) = e^{-r\tau}\left[F_{t,\tau}N(d_1) - XN(d_2)\right] \tag{58}$$

where $d_1 = \frac{\ln[F_{t,\tau}/X] + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}$, and $d_2 = d_1 - \sigma\sqrt{\tau}$.

Options on Futures

- Consider an option written on a futures price F_{t,t^*}, the date t futures price for a contract maturing at date t^*.
- The undiscounted profit (*loss*) earned by the long (*short*) party over the period from date t to date $T \leq t^*$ is simply $F_{T,t^*} - F_{t,t^*}$.
- Like forward contracts, there is no initial cost for the parties who enter into a futures contract. Hence, in a risk-neutral world, their expected profits must be zero:

$$\hat{E}_t \left[F_{T,t^*} - F_{t,t^*} \right] = 0 \tag{59}$$

so under the Q measure, the futures price is a martingale:

$$\hat{E}_t \left[F_{T,t^*} \right] = F_{t,t^*} \tag{60}$$

Options on Futures cont'd

- Since an asset's expected return under Q must be r, a futures price is like the price of an asset with a dividend yield of $\delta = r$.
- The value of a futures call option that matures in τ periods where $\tau \leq (t^* - t)$ is

$$c(t) = e^{-r\tau}\left[F_{t,t^*} N(d_1) - XN(d_2)\right] \qquad (61)$$

where $d_1 = \frac{\ln\left[F_{t,t^*}/X\right] + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}$, and $d_2 = d_1 - \sigma\sqrt{\tau}$.

- Note that this is similar in form to an option on a foreign currency written in terms of the forward exchange rate.

Term Structure Revisited

- Let $P(t,\tau)$ be the date t price of a default-free bond paying \$1 at maturity $T = t + \tau$.
- Interpreting $c(T) = P(T,0) = 1$, equation (17) is

$$P(t,\tau) = \widehat{E}_t \left[e^{-\int_t^T r(u)du} 1 \right] \tag{62}$$

- We now rederive the Vasicek (1977) model using this equation where recall that the physical process for $r(t)$ is

$$dr(t) = \alpha \left[\overline{r} - r(t) \right] dt + \sigma_r dz_r \tag{63}$$

- Assuming, like before, that the market price of bond risk q is a constant,

$$\mu_p(r,\tau) = r(t) + q\sigma_p(\tau) \tag{64}$$

where $\sigma_p(\tau) = -P_r \sigma_r / P$.

Term Structure cont'd

- Thus, recall that the physical process for a bond's price is

$$\begin{aligned} dP(r,\tau)/P(r,\tau) &= \mu_p(r,\tau)\,dt - \sigma_p(\tau)\,dz_r \qquad (65) \\ &= [r(t) + q\sigma_p(\tau)]\,dt - \sigma_p(\tau)\,dz_r \end{aligned}$$

- Defining $d\widehat{z}_r = dz_r - qdt$, equation (65) becomes

$$\begin{aligned} dP(t,\tau)/P(t,\tau) &= [r(t) + q\sigma_p(\tau)]\,dt - \sigma_p(\tau)\,[d\widehat{z}_r + qdt] \\ &= r(t)\,dt - \sigma_p(\tau)\,d\widehat{z}_r \qquad (66) \end{aligned}$$

which is the risk-neutral process for the bond price since all bonds have the expected rate of return r under the Q measure.

Term Structure cont'd

- Therefore, the process for $r(t)$ under the Q measure is found by also substituting $d\widehat{z}_r = dz_r - qdt$:

$$\begin{aligned} dr(t) &= \alpha\left[\overline{r} - r(t)\right]dt + \sigma_r\left[d\widehat{z}_r + qdt\right] \\ &= \alpha\left[\left(\overline{r} + \frac{q\sigma_r}{\alpha}\right) - r(t)\right]dt + \sigma_r d\widehat{z}_r \qquad (67) \end{aligned}$$

  which has the unconditional mean $\overline{r} + q\sigma_r/\alpha$.

- Thus, when evaluating equation (62)

$$P(t,\tau) = \widehat{E}_t\left[\exp\left(-\int_t^T r(u)\,du\right)\right]$$

  this expectation is computed assuming $r(t)$ follows the process in (67).

- Doing so leads to the same solution given in the previous chapter, equation (9.41) in the text.

Summary

- Martingale pricing is a generalization of risk-neutral pricing that is applicable in complete markets.
- With dynamically complete markets, the continuous-time state price deflator has an expected growth rate equal to minus the risk-free rate and a standard deviation equal to the market price of risk.
- Contingent claims valuation often can be simplified by an appropriate normalization of asset prices, deflating either by the price of a riskless *or* risky asset.
- Martingale pricing can be applied to options written on assets paying continuous, proportional dividends, as well as default-free bonds.