Dynamic Hedging and PDE Valuation

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Introduction

- Asset prices are modeled as following diffusion processes, permitting the possibility of continuous trading.

- This environment can allow a market with an underlying asset, a contingent claim, and the risk-free asset to be dynamically complete.

- We illustrate the Black-Scholes-Merton portfolio hedging argument that results in a partial differential equation for a contingent claim's price.

- Examples are

  - The Black-Scholes (1973) option pricing model.
  - The Vasicek (1977) equilibrium term structure model.
  - The Merton (1973b) stochastic interest rate option pricing model.

Portfolio Dynamics in Continuous Time

- The insight of Black and Scholes (1973) and Merton (1973) is that when assets follow diffusion processes, an option's payoff can be replicated by continuous trading in its underlying asset and a risk-free asset.
- Consider an investor who can trade in any n different assets whose prices follow diffusion processes. Define $S_i(t)$ as the price per share of asset i at date t, where $i = 1, ..., n$.
- The instantaneous rate of return on the ith asset is

$$dS_i(t) / S_i(t) \ = \ \mu_i \, dt \ + \ \sigma_i \, dz_i \qquad (1)$$

with expected rate of return and variance μ_i and σ_i^2.
- Let $F(t)$ be the net cash outflow per unit time from the portfolio at date t.

Portfolio Dynamics in Continuous Time cont'd

- First consider the analogous discrete-time dynamics where each discrete period is of length h.
- Let $w_i(t)$ be the number of shares held by the investor in asset i from date t to $t + h$.
- The date t portfolio value is denoted as $H(t)$ and equals the prior period's holdings at date t prices:

$$H(t) = \sum_{i=1}^{n} w_i(t - h)S_i(t) \qquad (2)$$

- The net cash outflow over the period is $F(t) h$ which must equal the net sales of assets:

$$-F(t) h = \sum_{i=1}^{n} [w_i(t) - w_i(t - h)] S_i(t) \qquad (3)$$

Portfolio Dynamics in Continuous Time cont'd

- To derive the limits of equations (2) and (3) as of date t and as $h \to 0$, convert backward differences, such as $w_i(t) - w_i(t - h)$, to forward differences. Update one period:

$$
\begin{aligned}
-F(t+h)\, h &= \sum_{i=1}^{n} [w_i(t+h) - w_i(t)]\, S_i(t+h) \\
&= \sum_{i=1}^{n} [w_i(t+h) - w_i(t)]\, [S_i(t+h) - S_i(t)] \\
&\quad + \sum_{i=1}^{n} [w_i(t+h) - w_i(t)]\, S_i(t) \qquad (4)
\end{aligned}
$$

and

Portfolio Dynamics in Continuous Time cont'd

$$H(t+h) = \sum_{i=1}^{n} w_i(t)\, S_i(t+h) \qquad (5)$$

- Taking the limits of (4) and (5) as $h \to 0$:

$$-F(t)\, dt = \sum_{i=1}^{n} dw_i(t)\, dS_i(t) + \sum_{i=1}^{n} dw_i(t)\, S_i(t) \qquad (6)$$

and

$$H(t) = \sum_{i=1}^{n} w_i(t)\, S_i(t) \qquad (7)$$

Portfolio Dynamics in Continuous Time cont'd

- Applying Itô's lemma to (7), the dynamics of the portfolio's value are

$$dH\left(t\right) = \sum_{i=1}^{n} w_i(t)\, dS_i(t) + \sum_{i=1}^{n} dw_i(t) S_i(t) + \sum_{i=1}^{n} dw_i(t)\, dS_i(t)$$
(8)

- Substituting (6) into (8), we obtain

$$dH\left(t\right) = \sum_{i=1}^{n} w_i(t)\, dS_i(t) \; - F\left(t\right)\, dt$$
(9)

Portfolio Dynamics in Continuous Time cont'd

- Thus, the value changes by capital gains income less net cash outflows.
- Substitute $dS_i(t)$ in (1) into (9),

$$
\begin{aligned}
dH(t) &= \sum_{i=1}^{n} w_i(t)\, dS_i(t) \; - \; F(t)\, dt \qquad (10) \\
&= \sum_{i=1}^{n} w_i(t)\, [\mu_i\, S_i dt \; + \; \sigma_i S_i\, dz_i] \; - \; F(t)\, dt
\end{aligned}
$$

- Define the proportion of $H(t)$ invested in asset i as $\omega_i(t) = w_i(t) S_i(t)/H(t)$, then (10) becomes

Portfolio Dynamics in Continuous Time cont'd

$$dH(t) = \sum_{i=1}^{n} \omega_i(t) H(t) [\mu_i dt + \sigma_i dz_i] - F(t) dt \qquad (11)$$

- Collecting terms in dt,

$$dH(t) = \left[\sum_{i=1}^{n} \omega_i(t) H(t) \mu_i - F(t)\right] dt + \sum_{i=1}^{n} \omega_i(t) H(t) \sigma_i dz_i$$
$$(12)$$

- Note from (7) that $\sum_{i=1}^{n} \omega_i(t) = 1$. Adding a riskfree asset that pays $r(t)$, so that its portfolio proportion is $1 - \sum_{i=1}^{n} \omega_i(t)$, we obtain

Portfolio Dynamics in Continuous Time cont'd

$$
\begin{aligned}
dH\left(t\right) \;=\; & \left[\sum_{i=1}^{n}\omega_{i}\left(t\right)\left(\mu_{i}-r\right)H\left(t\right)\,+rH\left(t\right)-F\left(t\right)\right]dt \\
& +\sum_{i=1}^{n}\omega_{i}\left(t\right)H\left(t\right)\sigma_{i}\,dz_{i}
\end{aligned}
\tag{13}
$$

which is a continuous-time formulation of wealth dynamics.

- Having derived the dynamics of an arbitrary portfolio, we now consider the Black-Scholes dynamic hedge portfolio that replicates contingent claims.

Black-Scholes Model Assumptions

- Let $S(t)$ be the date t price per share of a stock that follows the diffusion process

$$dS = \mu S\, dt + \sigma S\, dz \qquad (14)$$

with time-varying μ but constant σ. Let r be the constant rate of return on a risk-free investment $B(t)$:

$$dB = rBdt \qquad (15)$$

Next, let there be a European call option written on the stock whose date t value is $c(S, t)$. Its maturity value at date T is

$$c(S(T), T) = \max[0, S(T) - X] \qquad (16)$$

Black-Scholes Model

- Assume that $c(S, t)$ is twice-differentiable in S and once- in t. Itô's lemma states that the option's value follows the process

$$dc = \left[\frac{\partial c}{\partial S}\,\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\right]\,dt + \frac{\partial c}{\partial S}\,\sigma S\,dz \quad (17)$$

- Consider a self-financing $(F(t) = 0\ \forall t)$, zero net investment portfolio that is short one unit of the call option and hedged with the stock and risk-free asset.

- Zero net investment implies that the amount invested in the risk-free asset must be $B(t) = c(t) - w(t)S(t)$ where $w(t)$ is the number of shares of stock.

- Thus, the hedge portfolio $H(t)$ has instantaneous return

$$dH(t) = -dc(t) + w(t)\,dS(t) + [c(t) - w(t)S(t)]\,rdt \quad (18)$$

Black-Scholes Model

- Substituting (14) and (17) into (18), we obtain

$$
\begin{aligned}
dH\left(t\right) &= -\left[\frac{\partial c}{\partial S}\,\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\right]dt - \frac{\partial c}{\partial S}\,\sigma S\,dz \\
&\quad +w\left(t\right)\left(\mu S\,dt + \sigma S\,dz\right) + \left[c\left(t\right) - w\left(t\right)S\left(t\right)\right]rdt \quad (19)
\end{aligned}
$$

- Set $w\left(t\right) = \partial c/\partial S$ to hedge the return on the option. Then,

$$
\begin{aligned}
dH\left(t\right) &= -\left[\frac{\partial c}{\partial S}\,\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\right]dt - \frac{\partial c}{\partial S}\,\sigma S\,dz \\
&\quad +\frac{\partial c}{\partial S}\left(\mu S\,dt + \sigma S\,dz\right) + \left[c\left(t\right) - \frac{\partial c}{\partial S}S\left(t\right)\right]rdt \\
&= \left[-\frac{\partial c}{\partial t} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} + rc\left(t\right) - rS\left(t\right)\frac{\partial c}{\partial S}\right]dt \quad (20)
\end{aligned}
$$

Black-Scholes Model

- The portfolio is hedged, so $dH(t)$ is riskfree and must equal r. It is also costless, so $H(0) = 0$ and

$$dH(0) = rH(0)\,dt = r \times 0\,dt = 0 \qquad (21)$$

so $H(t) = 0\ \forall t$ and $dH(t) = 0\ \forall t$. This implies

$$\frac{\partial c}{\partial t} \;+\; \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} \;+\; r\,S\,\frac{\partial c}{\partial S} \;-\; r\,c \;=\; 0 \qquad (22)$$

- This partial differential equation has boundary condition

$$c(S(T),\,T) \;=\; \max[\,0,\,S(T) - X\,] \qquad (23)$$

Black-Scholes Formula (1973)

- The solution to (22) subject to (23) is

$$c(S(t),\, t) \;=\; S(t)\, N(d_1) \;-\; X\, e^{-r\,(T-t)}\, N(d_2) \qquad (24)$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(S(t)/X\right) + \left(r + \tfrac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad (25)\\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

and $N(\cdot)$ is the standard normal distribution function.

- Similar to the binomial model, (24) does not depend on μ, but only on $S(t)$ and σ.

Black-Scholes Formula (1973) cont'd

- From put-call parity, the value of a European put is

$$
\begin{aligned}
p(S(t),\, t) &= c(S(t),\, t) + X\, e^{-r\,(T-t)} - S(t) \quad (26) \\
&= X\, e^{-r\,(T-t)} N(-d_2) - S(t) N(-d_1)
\end{aligned}
$$

- Taking the partial derivatives of (24) and (26) gives the hedge ratios

$$
\frac{\partial c}{\partial S} = N(d_1) \tag{27}
$$

$$
\frac{\partial p}{\partial S} = -N(-d_1) \tag{28}
$$

which implies $0 < \partial c/\partial S < 1$ and $-1 < \partial p/\partial S < 0$.

Vasicek (1977) Model

- When the prices of default-free bonds depend on continuous-time stochastic processes, continuous trading and the no-arbitrage condition place restrictions on their prices.
- We now consider the Vasicek (1977) "one-factor" term structure model where uncertainty is determined by the yield on the shortest-maturity bond, $r(t)$.
- Define $P(t, \tau)$ as the date t price of a bond that makes a single payment of \$1 at date $T = t + \tau$. The rate of return on the bond is $\frac{dP(t,\tau)}{P(t,\tau)}$ and $P(t,0) = \$1$.
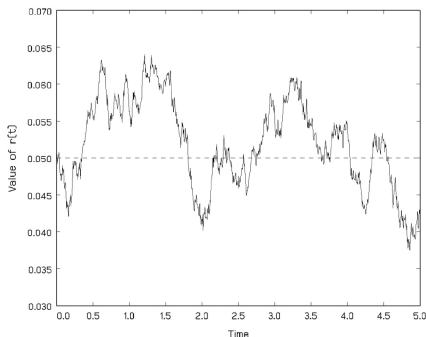- The instantaneous-maturity yield, $r(t)$, is defined as

$$\lim_{\tau \to 0} \frac{dP(t, \tau)}{P(t, \tau)} \equiv r(t)\, dt \qquad (29)$$

Process for $r(t)$

- $r(t)$ is assumed to follow the Ornstein-Uhlenbeck process:

$$dr(t) = \alpha \left[\overline{r} - r(t) \right] dt + \sigma_r dz_r \qquad (30)$$

where α, $\overline{r}$, and σ_r are positive constants. For $r(0) = \overline{r} = 0.05$, $\alpha = 0.3$, and $\sigma_r = 0.02$, a typical path is

Characteristics of O-U Process

- The Ornstein-Uhlenbeck process is reasonable for interest rates because it allows them to follow a mean-reverting stationary process.

- An end-of-chapter exercise shows that the instantaneous-maturity rate, $r(t)$, is normally distributed with

$$E_t\left[r(t+\tau)\right] = \overline{r} + e^{-\alpha\tau}\left(r(t) - \overline{r}\right) \qquad (31)$$

and

$$Var_t\left[r(t+\tau)\right] = \frac{\sigma_r^2}{2\alpha}\left(1 - e^{-2\alpha\tau}\right) \qquad (32)$$

- The parameter, α, measures how quickly $r(t)$ is expected to revert back to its unconditional value, $\overline{r}$.

Price Process

- Assume that bond prices of all maturities depend on only a single source of uncertainty $r(t)$; $P(r(t), \tau(t))$ where $\tau \equiv T - t$. Using Itô's lemma,

$$
\begin{aligned}
dP(r, \tau) &= \frac{\partial P}{\partial r} dr + \frac{\partial P}{\partial t} dt + \tfrac{1}{2} \frac{\partial^2 P}{\partial r^2} (dr)^2 \qquad (33) \\
&= \left[P_r \alpha(\bar{r} - r) + P_t + \tfrac{1}{2} P_{rr} \sigma_r^2 \right] dt + P_r \sigma_r dz_r \\
&= \mu_p(r, \tau) P(r, \tau) dt - \sigma_p(\tau) P(r, \tau) dz_r
\end{aligned}
$$

where subscripts on P denote partial derivatives and
$\mu_p(r, \tau) \equiv \frac{\left[P_r \alpha(\bar{r} - r) + P_t + \frac{1}{2} P_{rr} \sigma_r^2 \right]}{P(r, \tau)}$ and $\sigma_p(\tau) \equiv -\frac{P_r \sigma_r}{P(r, \tau)}$.

- Now make a portfolio containing one bond of maturity τ_1 and $-\frac{\sigma_p(\tau_1) P(r, \tau_1)}{\sigma_p(\tau_2) P(r, \tau_2)}$ units of a bond with maturity τ_2.

Hedge Portfolio

- Since both bond values depend on dz_r, the portfolio is hedged if we continually readjust the amount of the τ_2-maturity bond to equal $-\frac{\sigma_p(\tau_1)P(r,\tau_1)}{\sigma_p(\tau_2)P(r,\tau_2)}$ as $r(t)$ changes.
- The value of this hedge portfolio, $H(t)$, is

$$
\begin{aligned}
H(t) &= P(r,\tau_1) - \frac{\sigma_p(\tau_1)P(r,\tau_1)}{\sigma_p(\tau_2)P(r,\tau_2)}P(r,\tau_2) \quad (34) \\
&= P(r,\tau_1)\left[1 - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)}\right]
\end{aligned}
$$

and the hedge portfolio's instantaneous return is

$$
dH(t) = dP(r,\tau_1) - \frac{\sigma_p(\tau_1)P(r,\tau_1)}{\sigma_p(\tau_2)P(r,\tau_2)}dP(r,\tau_2) \quad (35)
$$

- Substituting for $dP(r,\tau_i)$ $i = 1,2$ from (33):

Hedge Portfolio cont'd

$$
\begin{aligned}
dH(t) &= \mu_p(r, \tau_1) P(r, \tau_1) dt - \sigma_p(\tau_1) P(r, \tau_1) dz_r \\
&\quad - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)} P(r, \tau_1) \mu_p(r, \tau_2) dt + \sigma_p(\tau_1) P(r, \tau_1) dz_r \\
&= \mu_p(r, \tau_1) P(r, \tau_1) dt - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)} P(r, \tau_1) \mu_p(r, \tau_2) dt
\end{aligned}
$$

- Since the portfolio return is riskless, its rate of return must equal the instantaneous riskless interest rate, $r(t)$:

$$
\begin{aligned}
dH(t) &= \left[\mu_p(r, \tau_1) - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)} \mu_p(r, \tau_2) \right] P(r, \tau_1) dt \quad (36) \\
&= r(t) H(t) dt = r(t) \left[1 - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)} \right] P(r, \tau_1) dt
\end{aligned}
$$

Bond Risk Premium

- The second line is from our definition of $H(t)$ in (34). Equating the two, we get the equality of bond Sharpe ratios:

$$\frac{\mu_p\left(r,\tau_1\right) - r\left(t\right)}{\sigma_p\left(\tau_1\right)} = \frac{\mu_p\left(r,\tau_2\right) - r\left(t\right)}{\sigma_p\left(\tau_2\right)} \qquad (37)$$

- Condition (37) requires all bonds to have a uniform market price of interest rate risk, as all risk is represented by dz_r.
- Cox, Ingersoll and Ross (1985a,b) derive this price of risk from general equilibrium, but for now we simply assume it is a constant q:

$$\frac{\mu_p\left(r,\tau\right) - r\left(t\right)}{\sigma_p\left(\tau\right)} = q \qquad (38)$$

or

$$\mu_p\left(r,\tau\right) = r\left(t\right) + q\sigma_p\left(\tau\right) \qquad (39)$$

Bond Risk Premium cont'd

- Substituting $\mu_p(r, \tau)$ and $\sigma_p(\tau)$ from (35) into (39):

$$P_r \alpha(\bar{r} - r) + P_t + \tfrac{1}{2} P_{rr} \sigma_r^2 = rP - q\sigma_r P_r \qquad (40)$$

which can be rewritten as

$$\frac{\sigma_r^2}{2} P_{rr} + (\alpha\bar{r} + q\sigma_r - \alpha r) P_r - rP + P_t = 0 \qquad (41)$$

- Since $d\tau = -dt$, so that $P_t \equiv \frac{\partial P}{\partial t} = -\frac{\partial P}{\partial \tau} \equiv -P_\tau$, equation (41) can be rewritten as

$$\frac{\sigma_r^2}{2} P_{rr} + [\alpha(\bar{r} - r) + q\sigma_r] P_r - rP - P_\tau = 0 \qquad (42)$$

subject to the boundary condition that at $\tau = 0$, $P(r, 0) = 1$.

Bond Risk Premium cont'd

- Equation (42) has a solution of the form

$$P\left(r\left(t\right),\tau\right) = A\left(\tau\right) e^{-B(\tau)r(t)} \qquad (43)$$

- Substituting back into (42) gives ordinary differential equations for A and B with boundary conditions $A\left(\tau = 0\right) = 1$ and $B\left(\tau = 0\right) = 0$ and solutions:

$$B\left(\tau\right) \equiv \frac{1 - e^{-\alpha\tau}}{\alpha} \qquad (44)$$

$$A\left(\tau\right) \equiv \exp\left[\left(B\left(\tau\right) - \tau\right)\left(\bar{r} + q\frac{\sigma_r}{\alpha} - \frac{1}{2}\frac{\sigma_r^2}{\alpha^2}\right) - \frac{\sigma_r^2 B\left(\tau\right)^2}{4\alpha}\right]$$

$$(45)$$

Characteristics of Bond Prices

- Using equation (43) in our definition of σ_p, we see that

$$\sigma_p\left(\tau\right) \equiv -\sigma_r \frac{P_r}{P} = \sigma_r B\left(\tau\right) = \frac{\sigma_r}{\alpha}\left(1 - e^{-\alpha\tau}\right) \quad (46)$$

which is an increasing and concave function of τ.

- Equation (39), $\mu_p\left(r,\tau\right) = r\left(t\right) + q\sigma_p\left(\tau\right)$, implies that a bond's expected rate of return increases (*decreases*) with its time until maturity if the market price of risk, q, is positive (*negative*).

Characteristics of Bond Prices cont'd

- A bond's continuously compounded yield, $Y(r(t), \tau)$, equals

$$
\begin{aligned}
Y(r(t), \tau) &= -\frac{1}{\tau} \ln\left[P(r(t), \tau)\right] \\
&= -\frac{1}{\tau} \ln\left[A(\tau)\right] + \frac{B(\tau)}{\tau} r(t) \qquad (47) \\
&= Y_\infty + \left[r(t) - Y_\infty\right] \frac{B(\tau)}{\tau} + \frac{\sigma_r^2 B(\tau)^2}{4\alpha\tau}
\end{aligned}
$$

where $Y_\infty \equiv \bar{r} + q\frac{\sigma_r}{\alpha} - \frac{1}{2}\frac{\sigma_r^2}{\alpha^2}$.

- Note that $\lim_{\tau \to \infty} Y(r(t), \tau) = Y_\infty$.

- Hence, the yield curve, which is the graph of $Y(r(t), \tau)$ as a function of τ, equals $r(t)$ at $\tau = 0$ and asymptotes to Y_∞ for τ large.

Bond Yield Slopes

- When $r(t) \leq Y_\infty - \frac{\sigma_r^2}{4\alpha^2} = \bar{r} + q\frac{\sigma_r}{\alpha} - \frac{3\sigma_r^2}{4\alpha^2}$, the yield curve is monotonically increasing.

- When $Y_\infty - \frac{\sigma_r^2}{4\alpha^2} < r(t) < Y_\infty + \frac{\sigma_r^2}{2\alpha^2} = \bar{r} + q\frac{\sigma_r}{\alpha}$, the yield curve has a humped shape.

- A monotonically downward sloping, or "inverted," yield curve occurs when $\bar{r} + q\frac{\sigma_r}{\alpha} \leq r(t)$.

- Since the yield curve is normally upward sloping, this suggests that $\bar{r} < \bar{r} + q\frac{\sigma_r}{\alpha} - \frac{3\sigma_r^2}{4\alpha^2}$, or $q > \frac{3\sigma_r}{4\alpha}$, i.e., a positive market price of bond risk.

Option Pricing with Random Interest Rates

- We now value contingent claims (first example) assuming stochastic interest rates (second example) to derive the Merton (1973b) option pricing model (third example).
- Define the price of a risk-free bond that pays $\$X$ at τ as $P(t, \tau) X$, so the option's value is $c(S(t), P(t, \tau), t)$.
- From Vasicek (1977), this bond's process is

$$dP(t, \tau) = \mu_p(t, \tau) P(t, \tau) dt + \sigma_p(\tau) P(t, \tau) dz_p \quad (48)$$

where from equation (33) define $dz_p \equiv -dz_r$ and assume a bond-stock correlation of $dz_p dz = \rho dt$. Applying Itô's lemma:

$$
\begin{aligned}
dc &= \left[\frac{\partial c}{\partial S} \mu S + \frac{\partial c}{\partial P} \mu_p P + \frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} \sigma^2 S^2 + \frac{1}{2} \frac{\partial^2 c}{\partial P^2} \sigma_p^2 P^2 \right. \\
&\quad \left. + \frac{\partial^2 c}{\partial S \partial P} \rho \sigma \sigma_p SP \right] dt + \frac{\partial c}{\partial S} \sigma S \, dz + \frac{\partial c}{\partial P} \sigma_p P \, dz_p \quad (49)
\end{aligned}
$$

Option Pricing with Random Interest Rates cont'd

$$= \mu_c c dt + \frac{\partial c}{\partial S} \sigma S \, dz + \frac{\partial c}{\partial P} \sigma_p P \, dz_p$$

where $\mu_c c$ is defined as the bracketed terms in (49).

- Our hedge portfolio is a unit short position in the option, a purchase of $w_s(t)$ units of the underlying stock, and a purchase of $w_p(t)$ units of the τ-maturity bond.

- A zero-net-investment restriction implies

$$c(t) - w_s(t) S(t) - w_p(t) P(t, \tau) = 0 \qquad (50)$$

- The hedge portfolio's return can then be written as

$$dH(t) = -dc(t) + w_s(t) \, dS(t) + w_p(t) \, dP(t, \tau) \qquad (51)$$

Hedge Portfolio with Random Interest Rates

$$
\begin{aligned}
= \quad & \left[-\mu_c c + w_s\left(t\right)\mu S + w_p\left(t\right)\mu_p P \right] dt \\
& + \left[-\frac{\partial c}{\partial S}\,\sigma S + w_s\left(t\right)\sigma S \right] dz \\
& + \left[-\frac{\partial c}{\partial P}\,\sigma_p P + w_p\left(t\right)\sigma_p P \right] dz_p \\
= \quad & \left[w_s\left(t\right)\left(\mu - \mu_c\right)S + w_p\left(t\right)\left(\mu_p - \mu_c\right)P \right] dt \\
& + \left[w_s\left(t\right) - \frac{\partial c}{\partial S} \right]\sigma S dz \\
& + \left[w_p\left(t\right) - \frac{\partial c}{\partial P} \right]\sigma_p P \, dz_p
\end{aligned}
$$

Hedge Portfolio with Random Interest Rates cont'd

- If $w_s(t)$ and $w_p(t)$ are chosen to make the portfolio's return riskless, then from (51) they must equal:

$$w_s(t) = \frac{\partial c}{\partial S} \qquad (52)$$

$$w_p(t) = \frac{\partial c}{\partial P} \qquad (53)$$

- But from the zero-net-investment condition (50), this can only be possible if it happens to be the case that

$$
\begin{aligned}
c &= w_s(t) S + w_p(t) P \\
&= S \frac{\partial c}{\partial S} + P \frac{\partial c}{\partial P} \qquad (54)
\end{aligned}
$$

Hedge Portfolio Dynamics

- By Euler's theorem, condition (54) holds if the option price is a homogeneous of degree 1 function of S and P.
- That is, $c\left(kS\left(t\right), kP\left(t, \tau\right), t\right) = kc\left(S\left(t\right), P\left(t, \tau\right), t\right)$. If so, then no-arbitrage implies $dH(t) = 0$:

$$w_s\left(t\right)\left(\mu - \mu_c\right)S + w_p\left(t\right)\left(\mu_p - \mu_c\right)P = 0 \qquad (55)$$

or

$$\frac{\partial c}{\partial S}\left(\mu - \mu_c\right)S + \frac{\partial c}{\partial P}\left(\mu_p - \mu_c\right)P = 0 \qquad (56)$$

which, using (54), can be rewritten as

$$\frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial P}\mu_p P - \mu_c c = 0 \qquad (57)$$

Hedge Portfolio Dynamics cont'd

- Substituting for $\mu_c c$ from (49), we obtain

$$-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2 - \frac{1}{2}\frac{\partial^2 c}{\partial P^2}\,\sigma_p^2 P^2 - \frac{\partial^2 c}{\partial S \partial P}\,\rho\sigma\sigma_p SP = 0 \quad (58)$$

which, since $\tau \equiv T - t$, can also be written as

$$\frac{1}{2}\left[\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2 + \frac{\partial^2 c}{\partial P^2}\,\sigma_p^2 P^2 + 2\frac{\partial^2 c}{\partial S \partial P}\,\rho\sigma\sigma_p SP\right] - \frac{\partial c}{\partial \tau} = 0$$
$$(59)$$

- The boundary conditions are $c\left(S\left(T\right), P\left(T, 0\right), T\right) = c\left(S\left(T\right), 1, T\right) = \max\left[S\left(T\right) - X, 0\right]$ where $P\left(t = T, \tau = 0\right) = 1$.
- The Merton (1973) solution is

Merton PDE Solution

$$c\left(S\left(t\right), P\left(t, \tau\right), \tau\right) = S(t)\,N(h_1) - P\left(t, \tau\right) X N(h_2) \qquad (60)$$

where

$$
\begin{aligned}
h_1 &= \frac{\ln\left(\frac{S(t)}{P(t,\tau)X}\right) + \frac{1}{2}v^2}{v} \\
h_2 &= h_1 - v
\end{aligned}
\qquad (61)
$$

where

$$v^2 = \int_0^\tau \left(\sigma^2 + \sigma_p\left(y\right)^2 - 2\rho\sigma\sigma_p\left(y\right)\right) dy \qquad (62)$$

- This is the Black-Scholes equation with v^2 replacing $\sigma^2\tau$. v^2 is the total variance of $\frac{S(t)}{P(t,\tau)X}$ from date t to date T, an interval of τ periods.

Merton PDE Solution cont'd

- If the bond's volatility is assumed to be that of the Vasicek model, $\sigma_p(y) = \frac{\sigma_r}{\alpha}(1 - e^{-\alpha y})$, then (62) is

$$
\begin{aligned}
v^2 &= \int_0^\tau \left(\sigma^2 + \frac{\sigma_r^2}{\alpha^2}\left(1 - 2e^{-\alpha y} + e^{-2\alpha y}\right) - 2\rho\sigma\frac{\sigma_r}{\alpha}\left(1 - e^{-\alpha y}\right) \right) dy \\
&= \sigma^2\tau + \frac{\sigma_r^2}{\alpha^3}\left(\alpha\tau + \frac{1 - e^{-2\alpha\tau}}{2} - 2\left(1 - e^{-\alpha\tau}\right) \right) \\
&\quad -2\rho\sigma\frac{\sigma_r}{\alpha^2}\left[\alpha\tau - \left(1 - e^{-\alpha\tau}\right) \right] \qquad (63)
\end{aligned}
$$

- Finally, note that the solution is homogeneous of degree 1 in $S(t)$ and $P(t, \tau)$, which verifies condition (54).

Summary

- When an underlying asset follows a diffusion and trade is can occur continuously, a portfolio can be created that fully hedges the risk of a contingent claim.

- In the absence of arbitrage, this hedge portfolio's return must equal the riskless rate, which implies an equilibrium partial differential equation for the contingent claim's value.

- This Black-Scholes-Merton hedging argument can derive values of options and determine a term structure of default-free interest rates.