Essentials of Diffusion Processes and Itô's Lemma

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Introduction

- We cover the basic properties of continuous-time stochastic processes having continuous paths, which are used to model many financial and economic time series.
- When asset prices follow such processes, dynamically complete markets may be possible when continuous trading is permitted.
- We show how:
    - A Brownian motion is a continuous-time limit of a discrete random walk.
    - Diffusion processes can be built from Brownian motions.
    - Itô's Lemma derives the process for a function of a variable that follows a continuous-time stochastic process.

Pure Brownian Motion

- Consider the stochastic process observed at date t, $z(t)$.
- Let Δt be a discrete change in time. The change in $z(t)$ over the time interval Δt is

$$z(t+\Delta t) - z(t) \equiv \Delta z = \sqrt{\Delta t}\,\tilde{\epsilon} \tag{1}$$

  where $\tilde{\epsilon}$ is a random variable with $E[\,\tilde{\epsilon}\,] = 0$, $Var[\,\tilde{\epsilon}\,] = 1$, and $Cov[\,z(t+\Delta t) - z(t),\, z(s+\Delta t) - z(s)\,] = 0$ if $(t,\, t+\Delta t)$ and $(s,\, s+\Delta t)$ are nonoverlapping time intervals.
- $z(t)$ is an example of a "random walk" process: $E[\Delta z] = 0$, $Var[\Delta z] = \Delta t$, and $z(t)$ has serially uncorrelated increments.
- Now consider the change in $z(t)$ over a fixed interval, from 0 to T. Assume T is made up of n intervals of length Δt.

Pure Brownian Motion cont'd

- Then

$$z(T) - z(0) = \sum_{i=1}^{n} \Delta z_i \tag{2}$$

where $\Delta z_i \equiv z(i \cdot \Delta t) - z([i-1] \cdot \Delta t) \equiv \sqrt{\Delta t}\,\tilde{\epsilon}_i$, and $\tilde{\epsilon}_i$ is the value of $\tilde{\epsilon}$ over the i^{th} interval. Hence (2) can be written

$$z(T) - z(0) = \sum_{i=1}^{n} \sqrt{\Delta t}\,\tilde{\epsilon}_i = \sqrt{\Delta t} \sum_{i=1}^{n} \tilde{\epsilon}_i \tag{3}$$

- Now the first two moments of $z(T) - z(0)$ are

$$E_0[z(T) - z(0)] = \sqrt{\Delta t} \sum_{i=1}^{n} E_0[\tilde{\epsilon}_i] = 0 \tag{4}$$

Continuous-Time Limit

$$Var_0[z(T) - z(0)] = \left(\sqrt{\Delta t}\right)^2 \sum_{i=1}^{n} Var_0[\tilde{\epsilon}_i] = \Delta t \cdot n \cdot 1 = T \tag{5}$$

where $E_t[\cdot]$ and $Var_t[\cdot]$ are conditional on information at date t.

- Given T, the mean and variance of $z(T) - z(0)$ are independent of n, the number of intervals.
- Keep T fixed but let $n \to \infty$. What do we know besides the first two moments? From the *Central Limit Theorem*,

$$\underset{n \to \infty}{p\lim}\,(z(T) - z(0)) = \underset{\Delta t \to 0}{p\lim}\,(z(T) - z(0)) \sim N(0,\ T)$$

Continuous-Time Limit cont'd

- Without loss of generality, assume $\tilde{\epsilon}_i \sim N(0,1)$. The limit of one of these minute independent increments can be defined as

$$dz(t) \equiv \lim_{\Delta t \to 0} \Delta z = \lim_{\Delta t \to 0} \sqrt{\Delta t}\tilde{\epsilon} \tag{6}$$

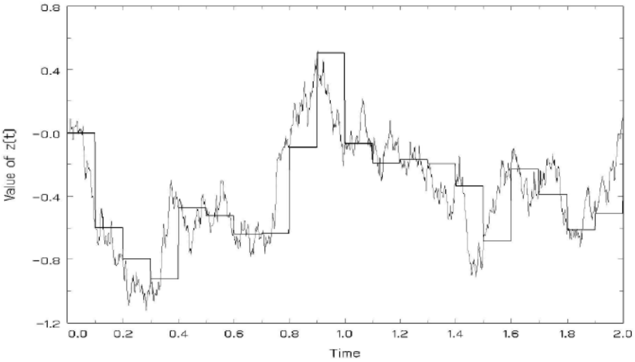
- Hence, $E[\,dz(t)\,] = 0$ and $Var[\,dz(t)\,] = dt$, i.e., the size of the time interval as $\Delta t \to 0$: $\int_0^T dt = T$.
- dz is referred to as a pure Brownian motion or Wiener process. It follows that

$$z(T) - z(0) = \int_0^T dz(t) \sim N(0,\ T) \tag{7}$$

- The integral in (7) is a *stochastic* or Itô integral.

Continuous-Time Limit cont'd

- $z(t)$ is a continuous process that is nowhere differentiable; $dz(t)/dt$ does not exist.
- Below is a $z(t)$ with $T = 2$ and $n = 20$, so that $\Delta t = 0.1$. As $n \to \infty$, so that $\Delta t \to 0$, $z(t)$ becomes Brownian motion.



Diffusion Processes

- Define a new process $x(t)$ by

$$dx(t) = \sigma\, dz(t) \tag{8}$$

- Then over a discrete interval, $[0, T]$, $x(t)$ is distributed

$$x(T) - x(0) = \int_0^T dx = \int_0^T \sigma\, dz(t) = \sigma \int_0^T dz(t) \sim N(0, \sigma^2 T) \tag{9}$$

- Next, add a deterministic (nonstochastic) change of $\mu(t)$ per unit of time to the $x(t)$ process:

$$dx = \mu(t) dt + \sigma dz \tag{10}$$

- Over any discrete interval, $[0, T]$, we obtain

Diffusion Processes cont'd

$$
\begin{aligned}
x(T) - x(0) &= \int_0^T dx = \int_0^T \mu(t)dt + \int_0^T \sigma\, dz(t) \qquad (11) \\
&= \int_0^T \mu(t)dt + \sigma \int_0^T dz(t) \sim N\left(\int_0^T \mu(t)dt,\, \sigma^2 T\right)
\end{aligned}
$$

- If $\mu(t) = \mu$, a constant, then $x(T) - x(0) = \mu T + \sigma \int_0^T dz(t) \sim N(\mu T, \sigma^2 T)$.
- The process $dx = \mu dt + \sigma dz$ is arithmetic Brownian motion.
- More generally, if μ and σ are functions of time, t, and/or $x(t)$, the *stochastic differential equation* describes $x(t)$

$$dx(t) = \mu[x(t), t]\, dt + \sigma[x(t), t]\, dz \qquad (12)$$

Diffusion Processes cont'd

- It is a continuous-time Markov process with drift $\mu[x(t), t]$ and volatility $\sigma[x(t), t]$.
- Equation (12) can be rewritten as an integral equation:

$$x(T) - x(0) = \int_0^T dx = \int_0^T \mu[x(t), t]\, dt + \int_0^T \sigma[x(t), t]\, dz \tag{13}$$

- $dx(t)$ is *instantaneously* normally distributed with mean $\mu[x(t), t]\, dt$ and variance $\sigma^2[x(t), t]\, dt$, but over any finite interval, $x(t)$ generally is not normally distributed.

Definition of an Itô Integral

- An Itô integral is formally defined as a mean-square limit of a sum involving the discrete Δz_i processes. For example, the Itô integral $\int_0^T \sigma[x(t),\ t]\, dz$, is defined from

$$\lim_{n\to\infty} E_0 \left[\left(\sum_{i=1}^{n} \sigma\left[x\left([i-1]\cdot\Delta t \right), [i-1]\cdot\Delta t \right] \Delta z_i - \int_0^T \sigma[x(t),\ t]\, dz \right)^2 \right] = 0 \tag{14}$$

  where within the parentheses of (14) is the difference between the Itô integral and its discrete-time approximation.

- An important Itô integral is $\int_0^T \left[dz\left(t\right) \right]^2$. In this case, (14) gives its definition

$$\lim_{n\to\infty} E_0 \left[\left(\sum_{i=1}^{n} \left[\Delta z_i \right]^2 - \int_0^T \left[dz\left(t\right) \right]^2 \right)^2 \right] = 0 \tag{15}$$

Definition of an Itô Integral cont'd

- To understand $\int_0^T [dz(t)]^2$, recall from (5) that

$$Var_0[z(T) - z(0)] = Var_0\left[\sum_{i=1}^{n} \Delta z_i\right] = E_0\left[\left(\sum_{i=1}^{n} \Delta z_i\right)^2\right]$$
$$= E_0\left[\sum_{i=1}^{n} [\Delta z_i]^2\right] = T \qquad (16)$$

because Δz_i are serially uncorrelated.

- One can show that

$$E_0\left[\left(\sum_{i=1}^{n} [\Delta z_i]^2 - T\right)^2\right] = 2T\Delta t \qquad (17)$$

Mean Square Convergence Proof

$$
\begin{aligned}
& E_0\left[\left(\sum_{i=1}^{n}[\Delta z_i]^2 - T\right)^2\right] = \\
= \quad & E_0\left[\sum_{i=1}^{n}[\Delta z_i]^2 \sum_{j=1}^{n}\left[\Delta z_j\right]^2\right] - 2E_0\left[\sum_{i=1}^{n}[\Delta z_i]^2\right] T + T^2 \\
= \quad & E_0\left[\sum_{i=1}^{n}[\Delta z_i]^4\right] + E_0\left[\sum_{i\neq j}^{n}[\Delta z_i]^2\left[\Delta z_j\right]^2\right] - 2T^2 + T^2 \\
= \quad & 3n(\Delta t)^2 + (n^2 - n)(\Delta t)^2 - T^2 = 3n(\Delta t)^2 - n(\Delta t)^2 + T^2 - T^2 \\
= \quad & 2(n\Delta t)\Delta t = 2T\Delta t
\end{aligned}
$$

- The limit as $\Delta t \to 0$, or $n \to \infty$, of (17) results in

$$\lim_{n\to\infty} E_0\left[\left(\sum\nolimits_{i=1}^{n}[\Delta z_i]^2 - T\right)^2\right] = \lim_{\Delta t\to 0} 2T\Delta t = 0 \qquad (18)$$

Convergence

- Comparing (15) with (18) implies that in mean-square convergence:

$$\int_0^T [dz(t)]^2 = T = \int_0^T dt \tag{19}$$

- Since $\int_0^T [dz(t)]^2$converges to $\int_0^T dt$ for any T, over an infinitesimally short time period $[dz(t)]^2$ converges to dt.
- If F is a function of the current value of a diffusion process, $x(t)$, and (possibly) also is a direct function of time, Itô's lemma shows us how to characterize $dF(x(t), t)$.

Functions of Continuous-Time Processes and Itô's Lemma

- Itô's lemma is the *fundamental theorem of stochastic calculus*.
- It derives the process of a function of a diffusion process.
- *Itô's Lemma* (*univariate case*): Let $x(t)$ follow the stochastic differential equation $dx(t) = \mu(x,t)\,dt + \sigma(x,t)\,dz$. Also let $F(x(t),\, t)$ be at least a twice-differentiable function. Then the differential of $F(x,\, t)$ is

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\,(dx)^2 \tag{20}$$

where the product $(dx)^2 = \sigma(x,t)^2 dt$. Hence, substituting in for dx and $(dx)^2$, (20) can be rewritten:

$$dF = \left[\frac{\partial F}{\partial x}\,\mu(x,t) + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\,\sigma^2(x,t)\right] dt + \frac{\partial F}{\partial x}\,\sigma(x,t)\,dz \tag{21}$$

Informal Proof

Proof: (See book for references to a formal proof, this is the intuition.)
Expand $F(x(t+\Delta t), t+\Delta t)$ in a Taylor series around t and $x(t)$:

$$\begin{aligned} F(x(t+\Delta t), t+\Delta t) &= F(x(t), t) + \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\left[\frac{\partial^2 F}{\partial x^2}(\Delta x)^2 \right. \\ &\quad \left. + 2\frac{\partial^2 F}{\partial x \partial t}\Delta x \Delta t + \frac{\partial^2 F}{\partial t^2}(\Delta t)^2\right] + H \end{aligned} \tag{22}$$

where $\Delta x \equiv x(t+\Delta t) - x(t)$ and H represents terms with higher orders of Δx and Δt. A discrete-time approximation of Δx can be written as

$$\Delta x = \mu(x,t)\,\Delta t + \sigma(x,t)\sqrt{\Delta t}\tilde{\epsilon} \tag{23}$$

Informal Proof cont'd

Defining $\Delta F \equiv F\left(x(t+\Delta t), t+\Delta t\right) - F(x\left(t\right),\ t)$ and substituting (23) in for Δx, equation (22) can be rewritten as

$$
\begin{aligned}
\Delta F \;=\; & \frac{\partial F}{\partial x}\left(\mu(x,t)\,\Delta t + \sigma(x,t)\sqrt{\Delta t}\tilde{\epsilon}\right) + \frac{\partial F}{\partial t}\Delta t \\
& + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left(\mu(x,t)\,\Delta t + \sigma(x,t)\sqrt{\Delta t}\tilde{\epsilon}\right)^2 \qquad (24) \\
& + \frac{\partial^2 F}{\partial x \partial t}\left(\mu(x,t)\,\Delta t + \sigma(x,t)\sqrt{\Delta t}\tilde{\epsilon}\right)\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}\left(\Delta t\right)^2 + H
\end{aligned}
$$

Consider the limit as $\Delta t \rightarrow dt$ and $\Delta F \rightarrow dF$. Recall from (6) that $\sqrt{\Delta t}\tilde{\epsilon}$ becomes dz and from (19) that $\left[\sqrt{\Delta t}\tilde{\epsilon}\right]\left[\sqrt{\Delta t}\tilde{\epsilon}\right]$ becomes $\left[dz\left(t\right)\right]^2 \rightarrow dt$. All terms of the form $dzdt \rightarrow 0$, and $dt^n \rightarrow 0$ as $\Delta t \rightarrow dt$ whenever $n > 1$.

Informal Proof cont'd

$$
\begin{aligned}
(dx)^2 &= (\mu(x,t)\,dt + \sigma(x,t)\,dz)^2 \qquad (25)\\
&= \mu(x,t)^2\,(dt)^2 + 2\mu(x,t)\sigma(x,t)dtdz + \sigma(x,t)^2\,(\,dz)^2\\
&= \sigma(x,t)^2\,(\,dz)^2 = \sigma(x,t)^2 dt
\end{aligned}
$$

So as $\Delta t \to dt$, $\sqrt{\Delta t}\tilde{\epsilon} \to dz$,

$$
\begin{aligned}
\Delta F &= \frac{\partial F}{\partial x}\left(\mu(x,t)\,\Delta t + \sigma(x,t)\sqrt{\Delta t}\tilde{\epsilon}\right) + \frac{\partial F}{\partial t}\Delta t\\
&+ \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left(\mu(x,t)\,\Delta t + \sigma(x,t)\sqrt{\Delta t}\tilde{\epsilon}\right)^2\\
&+ \frac{\partial^2 F}{\partial x \partial t}\left(\mu(x,t)\,\Delta t + \sigma(x,t)\sqrt{\Delta t}\tilde{\epsilon}\right)\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\Delta t)^2 + H
\end{aligned}
$$

becomes

$$
dF = \left[\frac{\partial F}{\partial x}\,\mu(x,t) + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\,\sigma^2(x,t)\right] dt + \frac{\partial F}{\partial x}\,\sigma(x,t)\,dz
$$

Geometric Brownian Motion

- Geometric Brownian motion is given by

$$dx = \mu x \, dt + \sigma x \, dz \tag{26}$$

  and is useful for modeling common stock prices since if x starts positive, it always remains positive (mean and variance are both proportional to its current value, x).
- Now consider $F(x,t) = \ln(x)$, (e.g., $dF = d(\ln x)$ is the rate of return). Applying Itô's lemma, we have

$$\begin{aligned} dF &= d(\ln x) = \left[\frac{\partial(\ln x)}{\partial x} \mu x + \frac{\partial(\ln x)}{\partial t} + \frac{1}{2} \frac{\partial^2(\ln x)}{\partial x^2} (\sigma x)^2 \right] dt \\ &\quad + \frac{\partial(\ln x)}{\partial x} \sigma x \, dz \\ &= \left[\mu + 0 - \frac{1}{2}\sigma^2 \right] dt + \sigma \, dz \end{aligned} \tag{27}$$

Geometric Brownian Motion cont'd

- Thus, $F = \ln x$ follows *arithmetic* Brownian motion. Since we know that

$$F(T) - F(0) \sim N\left((\mu - \frac{1}{2}\sigma^2)\,T,\, \sigma^2\,T\right) \qquad (28)$$

  then $x(t) = e^{F(t)}$ has a lognormal distribution over any discrete interval (by the definition of a lognormal random variable).

- Hence, geometric Brownian motion is lognormally distributed over any time interval.

Backward Kolmogorov Equation

- In general, finding the discrete-time distribution of a variable that follows a diffusion is useful for
  - computing its expected value
  - maximum likelihood estimation on discrete data
- Let $p(x, T; x_t, t)$ be the probability density function for diffusion x at date T given that it equals x_t at date t, where $T \geq t$. Applying Itô's lemma (assuming differentiability in t and twice- in x_t):

$$dp = \left[\frac{\partial p}{\partial x_t}\mu(x_t, t) + \frac{\partial p}{\partial t} + \frac{1}{2}\frac{\partial^2 p}{\partial x_t^2}\sigma^2(x_t, t)\right] dt + \frac{\partial p}{\partial x_t}\sigma(x_t, t)\, dz \tag{29}$$

- The expected change (i.e. drift) of p should be zero.

Backward Kolmogorov Equation cont'd

- Therefore,

$$\mu[x_t,\ t]\frac{\partial p}{\partial x_t}+\frac{\partial p}{\partial t}+\ \frac{1}{2}\sigma^2\left(x_t,t\right)\frac{\partial^2 p}{\partial x_t^2}=0 \qquad (30)$$

- Condition (30) is the backward Kolmogorov equation.
- This partial differential equation for $p\left(x,T;x_t,t\right)$ is solved subject to the boundary condition that when t becomes equal to T, then x must equal x_t with probability 1.
- Formally, this boundary condition is $p\left(x,T;x_T,T\right)=\delta\left(x-x_T\right)$, where $\delta\left(\cdot\right)$ is the Dirac delta function: $\delta\left(0\right)=\infty$, $\delta\left(y\right)=0\ \forall\ y\neq 0$, and $\int_{-\infty}^{\infty}\delta\left(y\right)dy=1$.

Backward Kolmogorov Equation cont'd

- Example: if $\mu[x_t,\ t] = \mu x_t,\ \sigma^2\left(x_t, t\right) = \sigma^2 x_t^2$ (geometric Brownian motion), the Kolmogorov equation is

$$\frac{1}{2}\sigma^2 x_t^2 \frac{\partial^2 p}{\partial x_t^2} + \mu x_t \frac{\partial p}{\partial x_t} + \frac{\partial p}{\partial t} = 0 \tag{31}$$

- Substituting into (31), it can be verified that the solution is

$$p\left(x, T, x_t, t\right) = \frac{1}{x\sqrt{2\pi\sigma^2\left(T-t\right)}} \exp\left[-\frac{\left(\ln x - \ln x_t - \left(\mu - \frac{1}{2}\sigma^2\right)\left(T-t\right)\right)^2}{2\sigma^2\left(T-t\right)}\right] \tag{32}$$

which is the lognormal probability density function for the random variable $x \in (0, \infty)$.

Multivariate Diffusions and Itô's Lemma

- Suppose there are m diffusion processes

$$dx_i = \mu_i \, dt + \sigma_i \, dz_i \qquad i = 1, \ldots, m, \tag{33}$$

and $dz_i dz_j = \rho_{ij} dt$, where ρ_{ij} is the correlation between Wiener process dz_i and dz_j.

- Recall that $dz_i dz_i = (dz_i)^2 = dt$. Now if dz_{iu} is uncorrelated with dz_i, dz_j can be written:

$$dz_j = \rho_{ij} dz_i + \sqrt{1 - \rho_{ij}^2} dz_{iu} \tag{34}$$

- Then from this interpretation of dz_j, we have

$$\begin{aligned} dz_j dz_j &= \rho_{ij}^2 (dz_i)^2 + \left(1 - \rho_{ij}^2\right) (dz_{iu})^2 + 2\rho_{ij} \sqrt{1 - \rho_{ij}^2} dz_i dz_{iu} \\ &= \rho_{ij}^2 dt + \left(1 - \rho_{ij}^2\right) dt + 0 \qquad (35) \\ &= dt \end{aligned}$$

Multivariate Itô's Lemma

and

$$\begin{aligned} dz_i dz_j &= dz_i \left(\rho_{ij} dz_i + \sqrt{1-\rho_{ij}^2} dz_{iu}\right) \\ &= \rho_{ij} (dz_i)^2 + \sqrt{1-\rho_{ij}^2} dz_i dz_{iu} \\ &= \rho_{ij} dt + 0 \end{aligned} \quad (36)$$

- Thus, ρ_{ij} can be interpreted as the proportion of dz_j that is perfectly correlated with dz_i.
- Let $F(x_1, \ldots, x_m, t)$ be at least a twice-differentiable function. Then the differential of $F(x_1, \ldots, x_m, t)$ is

$$dF = \sum_{i=1}^{m} \frac{\partial F}{\partial x_i} dx_i + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 F}{\partial x_i \partial x_j} dx_i \, dx_j \quad (37)$$

where $dx_i \, dx_j = \sigma_i \sigma_j \rho_{ij} \, dt$. Hence, (37) can be rewritten

Multivariate Itô's Lemma cont'd

$$dF = \left[\sum_{i=1}^{m}\left(\frac{\partial F}{\partial x_i}\,\mu_i + \frac{1}{2}\frac{\partial^2 F}{\partial x_i^2}\,\sigma_i^2\right) + \frac{\partial F}{\partial t} + \sum_{i=1}^{m}\sum_{j>i}^{m}\frac{\partial^2 F}{\partial x_i\,\partial x_j}\,\sigma_i\sigma_j\rho_{ij}\right] dt + \sum_{i=1}^{m}\frac{\partial F}{\partial x_i}\,\sigma_i\,dz_i \tag{38}$$

- Equation (38) generalizes Itô's lemma for a univariate diffusion, equation (21).
- Notably, the process followed by a function of several diffusion processes inherits each of the processes' Brownian motions.

Summary

- Brownian motion is the foundation of diffusion processes and is a continuous-time limit of a discrete-time random walk.
- Itô's lemma tells us how to find the process followed by a function of a diffusion process.
- The lemma can be used to derive the Kolmogorov equation, an important relation for finding the discrete-time distribution of a random variable that follows a diffusion process.
- The process followed by a function of several diffusions can be derived from a multivariate version of Itô's lemma.