Multiperiod Market Equilibrium

George Pennacchi

University of Illinois

Introduction

- The first order conditions from an individual's multiperiod consumption and portfolio choice problem can be interpreted as equilibrium conditions for asset pricing.
- A particular equilibrium asset pricing model where asset supplies are exogenous is the Lucas (1978) endowment economy.
- We also consider bubbles: nonfundamental asset price dynamics.

Asset Pricing in the Multiperiod Model

- In the multiperiod model, the individual's objective is

$$\max_{C_s,\{\omega_{is}\},\forall s,i} E_t \left[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T) \right] \qquad (1)$$

which is solved as a series of single period problems using the Bellman equation:

$$J(W_t, t) = \max_{C_t,\{\omega_{i,t}\}} U(C_t, t) + E_t\left[J(W_{t+1}, t+1)\right] \qquad (2)$$

- This led to the first-order conditions

$$\begin{aligned} U_C(C_t^*, t) &= R_{f,t} E_t\left[J_W(W_{t+1}, t+1)\right] \\ &= J_W(W_t, t) \end{aligned} \qquad (3)$$

$$E_t\left[R_{it} J_W(W_{t+1}, t+1)\right] = R_{f,t} E_t\left[J_W(W_{t+1}, t+1)\right], \;\; i = 1, ..., n \qquad (4)$$

Multiperiod Pricing Kernel

- This model has equilibrium implications even if assumptions about utility, income, and asset return distributions do not lead to explicit formulas for C_t^* and ω_{it}^*.
- Substituting the envelope condition $U_C\left(C_t^*, t\right) = J_W\left(W_t, t\right)$ at $t+1$ into the right-hand side of the first line of (3),

$$\begin{aligned} U_C\left(C_t^*, t\right) &= R_{f,t} E_t\left[J_W\left(W_{t+1}, t+1\right)\right] \\ &= R_{f,t} E_t\left[U_C\left(C_{t+1}^*, t+1\right)\right] \end{aligned} \qquad (5)$$

- Furthermore, substituting (4) into (3) and, again, using the envelope condition at date $t+1$ allows us to write

$$\begin{aligned} U_C\left(C_t^*, t\right) &= E_t\left[R_{it} J_W\left(W_{t+1}, t+1\right)\right] \\ &= E_t\left[R_{it} U_C\left(C_{t+1}^*, t+1\right)\right] \end{aligned} \qquad (6)$$

or

Multiperiod Pricing Kernel cont'd

$$\begin{aligned} 1 &= E_t \left[m_{t,t+1} R_{it} \right] \\ &= R_{f,t} E_t \left[m_{t,t+1} \right] \end{aligned} \tag{7}$$

where $m_{t,t+1} \equiv U_C \left(C^*_{t+1}, t+1 \right) / U_C \left(C^*_t, t \right)$ is the SDF (pricing kernel) between dates t and $t+1$.

- The relationship derived in the single-period context holds more generally: Updating equation (6) for risky asset j one period, $U_C \left(C^*_{t+1}, t+1 \right) = E_{t+1} \left[R_{j,t+1} U_C \left(C^*_{t+2}, t+2 \right) \right]$, and substituting in the right-hand side of the original (6), one obtains

$$\begin{aligned} U_C \left(C^*_t, t \right) &= E_t \left[R_{it} E_{t+1} \left[R_{j,t+1} U_C \left(C^*_{t+2}, t+2 \right) \right] \right] \\ &= E_t \left[R_{it} R_{j,t+1} U_C \left(C^*_{t+2}, t+2 \right) \right] \end{aligned} \tag{8}$$

Multiperiod Pricing Kernel cont'd

or

$$1 = E_t \left[R_{it} R_{j,t+1} m_{t,t+2} \right] \quad (9)$$

where $m_{t,t+2} \equiv U_C\left(C^*_{t+2}, t+2\right) / U_C\left(C^*_t, t\right)$ is the marginal rate of substitution, or the SDF, between dates t and $t+2$.

- By repeated substitution, (9) can be generalized to

$$1 = E_t \left[\mathrm{R}_{t,t+k} m_{t,t+k} \right] \quad (10)$$

where $m_{t,t+k} \equiv U_C\left(C^*_{t+k}, t+k\right) / U_C\left(C^*_t, t\right)$ and $\mathrm{R}_{t,t+k}$ is the return from any trading strategy involving multiple assets over the period from dates t to $t+k$.

- Equation (10) says that expected marginal utility-weighted returns are equal across all time periods and assets.
- These moment conditions are often tested using a Generalized Method of Moments technique.

Including Dividends in Asset Returns

- Let the return on the i^{th} risky asset, R_{it}, include a dividend payment made at date $t+1$, $d_{i,t+1}$, along with a capital gain, $P_{i,t+1} - P_{it}$:

$$R_{it} = \frac{d_{i,t+1} + P_{i,t+1}}{P_{it}} \tag{11}$$

- Substituting (11) into (7) and rearranging gives

$$P_{it} = E_t \left[\frac{U_C\left(C^*_{t+1}, t+1\right)}{U_C\left(C^*_t, t\right)} \left(d_{i,t+1} + P_{i,t+1}\right) \right] \tag{12}$$

- Similar to what was done in equation (8), substitute for $P_{i,t+1}$ using equation (12) updated one period to solve forward this equation.

Including Dividends in Asset Returns cont'd

$$
\begin{aligned}
P_{it} &= E_t\left[\frac{U_C\left(C^*_{t+1},t+1\right)}{U_C\left(C^*_t,t\right)}\left(d_{i,t+1}+\frac{U_C\left(C^*_{t+2},t+2\right)}{U_C\left(C^*_{t+1},t+1\right)}\left(d_{i,t+2}+P_{i,t+2}\right)\right)\right] \\
&= E_t\left[\frac{U_C\left(C^*_{t+1},t+1\right)}{U_C\left(C^*_t,t\right)}d_{i,t+1}+\frac{U_C\left(C^*_{t+2},t+2\right)}{U_C\left(C^*_t,t\right)}\left(d_{i,t+2}+P_{i,t+2}\right)\right] \qquad (13)
\end{aligned}
$$

- Repeating this type of substitution, that is, solving forward the difference equation (13), gives us

$$
P_{it} = E_t\left[\sum_{j=1}^{T}\frac{U_C\left(C^*_{t+j},t+j\right)}{U_C\left(C^*_t,t\right)}d_{i,t+j}+\frac{U_C\left(C^*_{t+T},t+T\right)}{U_C\left(C^*_t,t\right)}P_{i,t+T}\right] \qquad (14)
$$

where the integer T reflects a large number of future periods.

Including Time Preference δ

- If utility is of the form $U(C_t, t) = \delta^t u(C_t)$, where $\delta = \frac{1}{1+\rho} < 1$, then (14) becomes

$$P_{it} = E_t\left[\sum_{j=1}^{T} \delta^j \frac{u_C\left(C^*_{t+j}\right)}{u_C\left(C^*_t\right)} d_{i,t+j} + \delta^T \frac{u_C\left(C^*_{t+T}\right)}{u_C\left(C^*_t\right)} P_{i,t+T}\right] \quad (15)$$

- If individuals that have infinite lives or a bequest motive and $\lim_{T\to\infty} E_t\left[\delta^T \frac{u_C\left(C^*_{t+T}\right)}{u_C\left(C^*_t\right)} P_{i,t+T}\right] = 0$ (no speculative price "bubbles"), then

$$P_{it} = E_t\left[\sum_{j=1}^{\infty} \delta^j \frac{u_C\left(C^*_{t+j}\right)}{u_C\left(C^*_t\right)} d_{i,t+j}\right] \quad (16)$$

Dividends and Consumption

- In terms of the SDF $m_{t,\,t+j} \equiv \delta^j u_C\left(C^*_{t+j}\right)/u_C\left(C^*_t\right)$:

$$P_{it} = E_t\left[\sum_{j=1}^{\infty} m_{t,\,t+j}\, d_{i,t+j}\right] \tag{17}$$

- We next consider modeling the supply side of an economy to develop a theory of risky assets' dividends.

Lucas Model of Asset Pricing

- Lucas (1978) makes (17) into a general equilibrium model by assuming an *endowment* economy with infinitely-lived *representative* individuals.
- Risky asset i represents an ownership claim on an exogenous output process that pays a real dividend of d_{it} at date t.
- The dividend is nonstorable and non-reinvestable. With no wage income, aggregate consumption equals the total dividends paid by all of the n assets at that date:

$$C_t^* = \sum_{i=1}^{n} d_{it} \tag{18}$$

- Exogenous output implies that consumption and the stochastic discount factor are exogenous, which makes it easy to solve for the equilibrium prices.

Examples

- If the representative individual is risk-neutral, so that $u(C) = C$ and u_C is a constant (1), then (17) becomes

$$P_{it} = E_t\left[\sum_{j=1}^{\infty} \delta^j d_{i,t+j}\right] \tag{19}$$

- If utility is logarithmic $(u(C_t) = \ln C_t)$ and aggregate dividend $d_t = \sum_{i=1}^{n} d_{it}$, the price of risky asset i is given by

$$\begin{aligned} P_{it} &= E_t\left[\sum_{j=1}^{\infty} \delta^j \frac{C_t^*}{C_{t+j}^*} d_{i,t+j}\right] \\ &= E_t\left[\sum_{j=1}^{\infty} \delta^j \frac{d_t}{d_{t+j}} d_{i,t+j}\right] \end{aligned} \tag{20}$$

Examples cont'd

- Under logarithmic utility, we can price the market portfoio without making assumptions regarding the distribution of d_{it} in (20).
- If P_t is the value of aggregate dividends, then from (20):

$$\begin{aligned} P_t &= E_t\left[\sum_{j=1}^{\infty} \delta^j \frac{d_t}{d_{t+j}} d_{t+j}\right] \\ &= d_t \frac{\delta}{1-\delta} \end{aligned} \qquad (21)$$

- Note that higher expected future dividends, d_{t+j}, are exactly offset by a lower expected marginal utility of consumption, $m_{t,\,t+j} = \delta^j d_t / d_{t+j}$, leaving the value of a claim on this output process unchanged.

Examples cont'd

- For more general power utility, $u(C_t) = C_t^\gamma / \gamma$, we have

$$P_t = E_t\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{d_{t+j}}{d_t}\right)^{\gamma-1} d_{t+j}\right]$$
$$= d_t^{1-\gamma} E_t\left[\sum_{j=1}^{\infty} \delta^j d_{t+j}^{\gamma}\right] \quad (22)$$

  which does depend on the distribution of future dividends.

- The value of a hypothetical riskless asset that pays a one-period dividend of $1 is

$$P_{ft} = \frac{1}{R_{ft}} = \delta E_t\left[\left(\frac{d_{t+1}}{d_t}\right)^{\gamma-1}\right] \quad (23)$$

Examples cont'd

- We can view the Mehra and Prescott (1985) finding in its true multiperiod context: they used equations such as (22) and (23) with $d_t = C_t^*$ to see if a reasonable value of γ produces a risk premium (excess average return over a risk-free return) that matches that of market portfolio of U.S. stocks' historical average excess returns.
- Reasonable values of γ could not match the historical risk premium of 6 %, a result they described as the *equity premium puzzle*.
- As mentioned earlier, for reasonable levels of risk aversion, aggregate consumption appears to vary too little to justify the high Sharpe ratio for the market portfolio of stocks.
- The moment conditions in (22) and (23) require a highly negative value of γ to fit the data.

Labor Income

- We can add labor income to the market endowment (Cecchetti, Lam and Mark, 1993). Human capital pays a wage payment of y_t at date t, also non-storable. Hence, equilibrium aggregate consumption equals

$$C_t^* = d_t + y_t \tag{24}$$

so that equilibrium consumption no longer equals dividends.

- The value of the market portfolio is:

$$\begin{aligned} P_t &= E_t \left[\sum_{j=1}^{\infty} \delta^j \frac{u_C\left(C_{t+j}^*\right)}{u_C\left(C_t^*\right)} d_{t+j} \right] \\ &= E_t \left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C_{t+j}^*}{C_t^*} \right)^{\gamma-1} d_{t+j} \right] \end{aligned} \tag{25}$$

Labor Income cont'd

- Now specify separate lognormal processes for dividends and consumption:

$$\begin{aligned} \ln\left(C^*_{t+1}/C^*_t\right) &= \mu_c + \sigma_c \eta_{t+1} \\ \ln\left(d_{t+1}/d_t\right) &= \mu_d + \sigma_d \varepsilon_{t+1} \end{aligned} \quad (26)$$

where the error terms are serially uncorrelated and distributed as

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \quad (27)$$

- Now what is the equilibrium price of the market portfolio?

Labor Income cont'd

- When $\delta e^{\alpha} < 1$, the expectation in (25) equals

$$P_t = d_t \frac{\delta e^{\alpha}}{1 - \delta e^{\alpha}} \tag{28}$$

where

$$\alpha \equiv \mu_d - (1-\gamma)\mu_c + \frac{1}{2}\left[(1-\gamma)^2 \sigma_c^2 + \sigma_d^2\right] - (1-\gamma)\rho\sigma_c\sigma_d \tag{29}$$

(See Exercise 6.3.)

- Equation (28) equals (21) when $\gamma = 0$, $\mu_d = \mu_c$, $\sigma_c = \sigma_d$, and $\rho = 1$, which is the special case of log utility and no labor income.
- With no labor income ($\mu_d = \mu_c$, $\sigma_c = \sigma_d$, $\rho = 1$) but $\gamma \neq 0$, we have $\alpha = \gamma\mu_c + \frac{1}{2}\gamma^2\sigma_c^2$, which is increasing in the growth rate of dividends (and consumption) when $1 > \gamma > 0$.

Labor Income cont'd

- When $\gamma > 0$, greater dividend growth leads individuals to desire increased savings due to high intertemporal elasticity $(\varepsilon = 1/(1-\gamma) > 1)$. Market clearing requires the value of the market portfolio to rise, raising income or wealth to make desired consumption rise to equal the fixed supply.
- The reverse occurs when $\gamma < 0$, as the income or wealth effect will exceed the substitution effect.
- For the general case of labor income where α is given by equation (29), a lower correlation between consumption and dividends (decline in ρ) increases α.
- Since $\partial P_t / \partial \alpha > 0$, lower correlation raises the value of the market portfolio because it is a better hedge against uncertain labor income.

Rational Asset Price Bubbles

- Recall that an equilibrium asset pricing implication the individual's consumption and portfolio choices was equation (12):

$$P_{it} = E_t \left[\frac{U_C \left(C_{t+1}, t+1 \right)}{U_C \left(C_t, t \right)} \left(d_{i,t+1} + P_{i,t+1} \right) \right] \tag{30}$$

which for $U\left(C_t, t\right) = \delta^t u\left(C_t\right)$ is

$$P_{it} = E_t \left[\frac{\delta u_C \left(C_{t+1} \right)}{u_C \left(C_t \right)} \left(d_{i,t+1} + P_{i,t+1} \right) \right] \tag{31}$$

- Equation (31) can be rearranged as

$$E_t \left[P_{i,t+1} u_C \left(C_{t+1} \right) \right] = \delta^{-1} P_{it} u_C \left(C_t \right) - E_t \left[u_C \left(C_{t+1} \right) d_{i,t+1} \right] \tag{32}$$

Rational Asset Price Bubbles cont'd

- Now define $p_t \equiv P_{it} u_C(C_t)$, the product of the asset price and the marginal utility of consumption.

- Then equation (32) is

$$E_t[p_{t+1}] = \delta^{-1} p_t - E_t\left[u_C\left(C^*_{t+1}\right) d_{i,t+1}\right] \qquad (33)$$

where $\delta^{-1} = 1 + \rho > 1$ where ρ is the subjective rate of time preference.

Rational Asset Price Bubbles cont'd

- Under the assumption that

$$\lim_{T \to \infty} E_t \left[\delta^T u_c \left(C_{t+T} \right) P_{i,t+T} \right] = \lim_{T \to \infty} E_t \left[\delta^T p_T \right] = 0 \quad (34)$$

we obtained a solution to this equation given by equation (17) which we can call the *fundamental solution*, f_t:

$$p_t = f_t \equiv E_t \left[\sum_{j=1}^{\infty} \delta^j u_C \left(C_{t+j} \right) d_{i,t+j} \right] \quad (35)$$

- The sum in (35) converges if the marginal utility-weighted dividends are expected to grow more slowly than the time preference discount factor.

Rational Asset Price Bubbles cont'd

- There are other solutions to (33) of the form $p_t = f_t + b_t$ where the *bubble* component b_t is any process that satisfies

$$E_t\left[b_{t+1}\right] = \delta^{-1} b_t = \left(1+\rho\right) b_t \qquad (36)$$

- This is easily verified by substitution into (33):

$$\begin{aligned} E_t\left[f_{t+1} + b_{t+1}\right] &= \delta^{-1}\left(f_t + b_t\right) - E_t\left[u_C\left(C^*_{t+1}\right) d_{i,t+1}\right] \\ E_t\left[f_{t+1}\right] + E_t\left[b_{t+1}\right] &= \delta^{-1} f_t + \delta^{-1} b_t - E_t\left[u_C\left(C^*_{t+1}\right) d_{i,t+1}\right] \\ E_t\left[b_{t+1}\right] &= \delta^{-1} b_t = \left(1+\rho\right) b_t \qquad (37) \end{aligned}$$

where in the last line of (37) uses the fact that f_t satisfies the difference equation. Since $\delta^{-1} > 1$, b_t explodes in expected value:

Bubble Examples

$$\lim_{i\to\infty} E_t\left[b_{t+i}\right] = \lim_{i\to\infty}(1+\rho)^i b_t = \begin{cases} +\infty \text{ if } b_t > 0 \\ -\infty \text{ if } b_t < 0 \end{cases} \tag{38}$$

- The exploding nature of b_t provides a rationale for interpreting the general solution $p_t = f_t + b_t$, $b_t \neq 0$, as a bubble solution.
- Suppose that b_t follows a deterministic time trend:

$$b_t = b_0 (1+\rho)^t \tag{39}$$

- Then if $b_0 > 0$, the solution

$$p_t = f_t + b_0 (1+\rho)^t \tag{40}$$

implies that the marginal utility-weighted asset price grows exponentially forever.

Bubble Examples cont'd

- Next, consider a possibly more realistic modeling of a "bursting" bubble proposed by Blanchard (1979):

$$b_{t+1} = \begin{cases} \frac{1+\rho}{q} b_t + e_{t+1} & \text{with probability } q \\ z_{t+1} & \text{with probability } 1-q \end{cases} \tag{41}$$

with $E_t\left[e_{t+1}\right] = E_t\left[z_{t+1}\right] = 0$.

- The bubble continues with probability q each period but "bursts" with probability $1-q$.
- This process satisfies the condition in (36), so that $p_t = f_t + b_t$ is again a valid bubble solution, and the expected return conditional on no crash is higher than in the infinite bubble.

Likelihood of Rational Bubbles

- Additional economic considerations may rule out many rational bubbles: consider negative bubbles where $b_t < 0$.
- From (38) individuals must expect that at some future date $\tau > t$ that $p_\tau = f_\tau + b_\tau$ can be negative.
- Since marginal utility is always positive, $P_{it} = p_t / u_C \left(C_t \right)$ must be expected to become negative, which is inconsistent with limited-liability securities.

Likelihood of Rational Bubbles cont'd

- Now note that, in general, the bubble component satisfying $E_t[b_{t+1}] = \delta^{-1} b_t = (1+\rho) b_t$ can be rewritten as

$$b_{t+1} = (1+\rho) b_t + \varepsilon_{t+1} \tag{42}$$

where $E_t[\varepsilon_{t+1}] = 0$.

- This process implies

$$b_t = (1+\rho)^t b_0 + \sum_{s=1}^{t} (1+\rho)^{t-s} \varepsilon_s \tag{43}$$

where ε_s, $s = 1, ..., t$ are each mean-zero innovations.

- An implication is that bubbles that burst and start again can be ruled out. Why?

Likelihood of Rational Bubbles cont'd

- To avoid negative values of b_t (and negative expected future prices), realizations of ε_t must satisfy

$$\varepsilon_t \geq -(1+\rho)\, b_{t-1}, \;\; \forall t \geq 0 \tag{44}$$

- This is due to

$$\begin{aligned} \varepsilon_t &= b_t - (1+\rho)\, b_{t-1} \\ b_t &= \varepsilon_t + (1+\rho)\, b_{t-1} > 0 \\ \varepsilon_t &\geq -(1+\rho)\, b_{t-1} \end{aligned}$$

- For example, if $b_t = 0$ so that a bubble does not exist at date t, then from (44) and the requirement that ε_{t+1} have mean zero, it must be the case that $\varepsilon_{t+1} = 0$ with probability 1.
- Hence, if a bubble currently does not exist, it cannot get started.

Likelihood of Rational Bubbles cont'd

- Moreover, the bursting and then restarting bubble in (41) could only avoid a negative value of b_{t+1} if $z_{t+1} = 0$ with probability 1 and $e_{t+1} = 0$ whenever $b_t = 0$. Hence, this type of bubble would need to be positive on the first trading day, and once it bursts it could never restart.
- Tirole (1982) considers an economy model with a finite number of agents and shows that rational individuals will not trade assets at prices above their fundamental values.
- Santos and Woodford (1997) consider rational bubbles in a wide variety of economies and find only a few examples of the overlapping generations type where they can exist.
- If conditions for rational bubbles are limited yet bubbles seem to occur, some irrationality may be required.

Summary

- If an asset's dividends are modeled explicitly, the asset's price satisfies a discounted dividend formula.
- The Lucas endowment economy takes this a step further by equating aggregate dividends to consumption, simplifying valuation of claims on aggregate dividends.
- In an infinite horizon model, rational asset price bubbles are possible but additional aspects of the economic environment can often rule them out.