# Consumption-Savings Decisions and State Pricing

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# Introduction

- We now consider a consumption-savings decision along with the previous portfolio choice decision.
- These decisions imply a *stochastic discount factor* (SDF) based on marginal utilities of consumption at different dates.
- This SDF can value all traded assets and can bound assets' expected returns and volatilities.
- The SDF can also be derived by assuming *market completeness* and no arbitrage.
- We can modify the SDF to value assets using *risk-neutral probabilities*.

# Consumption and Porftolio Choices

- Let $W_0$ and $C_0$ be an individual's initial date 0 wealth and consumption, respectively. At date 1, the individual consumes all of his wealth $C_1$.
- The individual's utility function is:

$$U(C_0) + \delta E\left[U\left(\widetilde{C}_1\right)\right] \tag{1}$$

where $\delta = \frac{1}{1+\rho}$ is a subjective discount factor. A rate of time preference $\rho > 0$ reflects impatience for consuming early.

- There are $n$ assets where $P_i$ is the date 0 price per share and $X_i$ is the date 1 random payoff of asset $i$, $i = 1, ..., n$. Hence $R_i \equiv X_i / P_i$ is asset $i$'s random return.

# Consumption and Porftolio Choices cont'd

- The individual receives labor income of $y_0$ at date 0 and random labor income of $y_1$ at date 1.
- Let $\omega_i$ be the proportion of date 0 savings invested in asset $i$. Then the individual's intertemporal budget constraint is

$$C_1 = y_1 + (W_0 + y_0 - C_0) \sum_{i=1}^{n} \omega_i R_i \tag{2}$$

where $(W_0 + y_0 - C_0)$ is date 0 savings. The individual's maximization problem is

$$\max_{C_0, \{\omega_i\}} U(C_0) + \delta E[U(C_1)] \tag{3}$$

subject to $\sum_{i=1}^{n} \omega_i = 1$. Substituting in (2), the first-order conditions wrt $C_0$ and $\omega_i$, $i = 1, ..., n$ are

# Consumption and Porftolio Choices cont'd

$$U'\left(C_0\right) - \delta E\left[U'\left(C_1\right) \sum_{i=1}^{n} \omega_i R_i\right] = 0 \tag{4}$$

$$\delta E\left[U'\left(C_1\right) R_i\right] - \lambda = 0, \quad i = 1, ..., n \tag{5}$$

where $\lambda \equiv \lambda' / \left(W_0 + y_0 - C_0\right)$ and $\lambda'$ is the Lagrange multiplier for the constraint $\sum_{i=1}^{n} \omega_i = 1$.

- From (5), for any two assets $i$ and $j$:

$$E\left[U'\left(C_1\right) R_i\right] = E\left[U'\left(C_1\right) R_j\right] \tag{6}$$

- Equation (6) implies that the investor's optimal portfolio choices are such that the expected marginal utility-weighted returns of any two assets are equal.

# Consumption and Porftolio Choices cont'd

- How do we interpret equation (6)? Note that when $C_1$ is high, $U'(C_1)$ is low due to the concavity of utility.
- Thus, an asset that pays high returns when consumption is high (*low*) will be weighted by a low (*high*) marginal utility weight.
- $E[U'(C_1)R_i] = E[U'(C_1)R_j]$ implies diversification. Why?
- If the individual invests a lot in asset $i$, say $\omega_i = 1$ and $\omega_j = 0$, $j \neq i$, then $C_1 = y_1 + (W_0 + y_0 - C_0)R_i$.

# Consumption and Porftolio Choices cont'd

- Thus, $C_1$ and $R_i$ will be highly positively correlated and

$$E\left[U'\left(C_1\right)R_i\right] = E\left[U'\left(C_1\right)\right]E\left[R_i\right] + Cov\left(U'\left(C_1\right), R_i\right) \quad (7)$$

  will be low due to $Cov\left(U'\left(C_1\right), R_i\right) < 0$ while $E\left[U'\left(C_1\right)R_j\right]$ for other assets will tend to be high, implying $E\left[U'\left(C_1\right)R_i\right] < E\left[U'\left(C_1\right)R_j\right]$.

- Hence, to make $E\left[U'\left(C_1\right)R_i\right] = E\left[U'\left(C_1\right)R_j\right]$, the individual will re-allocate some savings from asset $i$ to make $C_1$ less correlated with asset $i$ and more correlated with the other assets.

# Consumption and Porftolio Choices cont'd

- To examine the optimal intertemporal allocation of resources, substitute (5) into (4)

$$\begin{aligned} U'\left(C_0\right) &= \delta E\left[U'\left(C_1\right)\sum_{i=1}^{n}\omega_i R_i\right] = \sum_{i=1}^{n}\omega_i \delta E\left[U'\left(C_1\right)R_i\right] \\ &= \sum_{i=1}^{n}\omega_i \lambda = \lambda \end{aligned} \quad (8)$$

- Then substituting $\lambda = U'\left(C_0\right)$ into (5) gives

$$\delta E\left[U'\left(C_1\right)R_i\right] = U'\left(C_0\right), \quad i = 1, ..., n \quad (9)$$

or, since $R_i = X_i/P_i$,

$$P_i U'\left(C_0\right) = \delta E\left[U'\left(C_1\right)X_i\right], \quad i = 1, ..., n \quad (10)$$

# Consumption and Porftolio Choices cont'd

- Equation (10) implies the individual invests in asset $i$ until the marginal utility of giving up $P_i$ dollars at date 0 just equals the expected marginal utility of receiving $X_i$ at date 1.
- Equation (10) for a risk-free asset that pays $R_f$ (gross return) is

$$U'(C_0) = R_f \delta E\left[U'(C_1)\right] \tag{11}$$

- With CRRA utility $U(C) = C^\gamma/\gamma$, for $\gamma < 1$, equation (11) is

$$\frac{1}{R_f} = \delta E\left[\left(\frac{C_0}{C_1}\right)^{1-\gamma}\right] \tag{12}$$

implying that when the interest rate is high, so is the expected growth in consumption.

# Consumption and Porftolio Choices cont'd

- If there is only a risk-free asset and nonrandom labor income, so that $C_1$ is nonstochastic, equation (12) is

$$R_f = \frac{1}{\delta}\left(\frac{C_1}{C_0}\right)^{1-\gamma} \tag{13}$$

- Note that

$$\begin{aligned} \frac{\partial R_f}{\partial \frac{C_1}{C_0}} &= \frac{1-\gamma}{\delta}\left(\frac{C_1}{C_0}\right)^{-\gamma} \\ &= (1-\gamma)\frac{R_f}{\frac{C_1}{C_0}} \end{aligned} \tag{14}$$

# Intertemporal Elasticity

- So that the intertemporal elasticity of substitution is

$$\epsilon \equiv \frac{R_f}{\frac{C_1}{C_0}} \frac{\partial \frac{C_1}{C_0}}{\partial R_f} = \frac{\partial \ln \left( C_1 / C_0 \right)}{\partial \ln \left( R_f \right)} = \frac{1}{1 - \gamma} \tag{15}$$

- Thus for CRRA utility, $\epsilon$ is the reciprocal of the coefficient of relative risk aversion. When $0 < \gamma < 1$, $\epsilon$ exceeds unity and a higher interest rate raises second-period consumption more than one-for-one.
- Conversely, when $\gamma < 0$, then $\epsilon < 1$ and a higher interest rate raises second-period consumption less than one-for-one, implying a decrease in initial savings.

# Intertemporal Elasticity cont'd

- The individual's response reflects two effects from an increase in interest rates.
    1. A *substitution effect* raises the return from transforming current consumption into future consumption, providing an incentive to save more.
    2. An *income effect* from the higher return on a given amount of savings makes the individual better off and, ceteris paribus, would raise consumption in both periods.
- For $\epsilon > 1$, the substitution effect outweighs the income effect, while the reverse occurs when $\epsilon < 1$. When $\epsilon = 1$, the income and substitution effects exactly offset each other.

# Equilibrium Asset Pricing Implications

- The individual's consumption - portfolio choice has asset pricing implications. Rewrite equation (10):

$$\begin{aligned} P_i &= E\left[\frac{\delta U'(C_1)}{U'(C_0)} X_i\right] \\ &= E\left[m_{01} X_i\right] \end{aligned} \tag{16}$$

  where $m_{01} \equiv \delta U'(C_1)/U'(C_0)$ is the *stochastic discount factor* or *state price deflator* for valuing asset returns.

- In states of nature where $C_1$ is high (due to high portfolio returns or high labor income), marginal utility, $U'(C_1)$, is low and an asset's payoffs are not highly valued.
- Conversely, in states where $C_1$ is low, marginal utility is high and an asset's payoffs are much desired.

# Stochastic Discount Factor

- The SDF or "*pricing kernel*" may differ across investors due to differences in random labor income that causes the distribution of $C_1$, and hence $\delta U'(C_1)/U'(C_0)$, to differ.
- Nonetheless, $E[m_{01}X_i] = E[\delta U'(C_1)X_i/U'(C_0)]$ is the same for all investors who can trade in asset $i$ since individuals adjust their portfolios to hedge individual-specific risks, and differences in $\delta U'(C_1)/U'(C_0)$ reflect only risks uncorrelated with asset returns.
- Utility depends on real consumption, $C_1$. If $P_i^N$ and $X_i^N$ are the initial price and end-of-period payoff measured in currency units (nominal terms), they need to be deflated by a price index to convert them to real quantities.

# Real Pricing Kernel

- Let $CPI_t$ be the consumer price index at date $t$. Equation (16) becomes

$$\frac{P_i^N}{CPI_0} = E\left[\frac{\delta U'(C_1)}{U'(C_0)} \frac{X_i^N}{CPI_1}\right] \tag{17}$$

- If we define $I_{ts} = CPI_s / CPI_t$ as 1 plus the inflation rate between dates $t$ and $s$, equation (17) is

$$\begin{aligned} P_i^N &= E\left[\frac{1}{I_{01}} \frac{\delta U'(C_1)}{U'(C_0)} X_i^N\right] \\ &= E\left[M_{01} X_i^N\right] \end{aligned} \tag{18}$$

where $M_{01} \equiv (\delta / I_{01}) U'(C_1) / U'(C_0)$ is the SDF for nominal returns, equal to the real pricing kernel, $m_{01}$, discounted at the (random) rate of inflation between dates 0 and 1.

# Risk Premia and Marginal Utility of Consumption

- Equation (16) can be rewritten to shed light on an asset's risk premium. Divide each side of (16) by $P_i$:

$$\begin{aligned} 1 &= E\left[m_{01} R_i\right] \qquad (19) \\ &= E\left[m_{01}\right] E\left[R_i\right] + Cov\left[m_{01}, R_i\right] \\ &= E\left[m_{01}\right] \left( E\left[R_i\right] + \frac{Cov\left[m_{01}, R_i\right]}{E\left[m_{01}\right]} \right) \end{aligned}$$

- Recall from (11) that for the case of a risk-free asset, $E\left[\delta U'\left(C_1\right)/U'\left(C_0\right)\right] = E\left[m_{01}\right] = 1/R_f$. Then (19) can be rewritten

$$R_f = E\left[R_i\right] + \frac{Cov\left[m_{01}, R_i\right]}{E\left[m_{01}\right]} \qquad (20)$$

or

# Risk Premia and Marginal Utility of Consumption cont'd

$$\begin{aligned} E[R_i] &= R_f - \frac{Cov[m_{01}, R_i]}{E[m_{01}]} \\ &= R_f - \frac{Cov[U'(C_1), R_i]}{E[U'(C_1)]} \end{aligned} \tag{21}$$

- An asset that tends to pay high returns when consumption is high (*low*) has $Cov[U'(C_1), R_i] < 0$ ($Cov[U'(C_1), R_i] > 0$) and will have an expected return greater (*less*) than the risk-free rate.
- Investors are satisfied with negative risk premia when assets hedge against low consumption states of the world.

# Relationship to the CAPM

- Suppose there is a portfolio with a random return of $\widetilde{R}_m$ that is perfectly negatively correlated with the marginal utility of date 1 consumption, $U'\left(\widetilde{C}_1\right)$, so that it is also perfectly negatively correlated with $m_{01}$:

$$U'(\tilde{C}_1) = \kappa_0 - \kappa\,\widetilde{R}_m, \quad \kappa_0 > 0, \quad \kappa > 0 \qquad (22)$$

- Then this implies

$$Cov[U'(C_1),\, R_m] = -\kappa\, Cov[R_m, R_m] = -\,\kappa\, Var[R_m] \qquad (23)$$

and

$$Cov[U'(C_1),\, R_i] = -\kappa\, Cov[R_m,\, R_i] \qquad (24)$$

# Relationship to the CAPM cont'd

- From (21), the risk premium on this portfolio is

$$E[R_m] = R_f - \frac{Cov[U'(C_1), R_m]}{E[U'(C_1)]} = R_f + \frac{\kappa Var[R_m]}{E[U'(C_1)]} \quad (25)$$

- Using (21) and (25) to substitute for $E[U'(C_1)]$, and using (24), we obtain

$$\frac{E[R_m] - R_f}{E[R_i] - R_f} = \frac{\kappa Var[R_m]}{\kappa Cov[R_m, R_i]} \quad (26)$$

and rearranging:

$$E[R_i] - R_f = \frac{Cov[R_m, R_i]}{Var[R_m]} (E[R_m] - R_f) \quad (27)$$

# Relationship to the CAPM cont'd

- Equation (27) is the CAPM relation

$$E[R_i] = R_f + \beta_i \left(E[R_m] - R_f\right) \tag{28}$$

- Note that CAPM assumptions imply the market portfolio is perfectly positively (*negatively*) correlated with consumption (*marginal utility of consumption*).
    1. There is no wage income, so end of period consumption derives only from asset portfolio returns.
    2. With a risk-free asset and normally distributed asset returns, everyone holds the same risky asset (market) portfolio.
- Hence, the only risk to $C_1$ is the return on the market portfolio.

# Bounds on Risk Premia

- $m_{01} \equiv \delta U'(C_1)/U'(C_0)$ places a bound on the Sharpe ratio of all assets. Rewrite equation (21) as

$$E[R_i] = R_f - \rho_{m_{01},R_i} \frac{\sigma_{m_{01}}\sigma_{R_i}}{E[m_{01}]} \tag{29}$$

where $\sigma_{m_{01}}$, $\sigma_{R_i}$, and $\rho_{m_{01},R_i}$ are the standard deviation of the discount factor, the standard deviation of the return on asset $i$, and the correlation between the discount factor and the return on asset $i$, respectively.

- Rearranging (29) leads to

$$\frac{E[R_i] - R_f}{\sigma_{R_i}} = -\rho_{m_{01},R_i} \frac{\sigma_{m_{01}}}{E[m_{01}]} \tag{30}$$

# Hansen-Jagannathan Bounds

- Since $-1 \leq \rho_{m_{01},R_i} \leq 1$, we know that

$$\left| \frac{E\left[R_i\right] - R_f}{\sigma_{R_i}} \right| \leq \frac{\sigma_{m_{01}}}{E\left[m_{01}\right]} = \sigma_{m_{01}} R_f \tag{31}$$

- Equation (31) was derived by Robert Shiller (1982) and generalized by Hansen and Jagannathan (1991).
- If there exists a portfolio of assets whose return is perfectly negatively correlated with $m_{01}$, then (31) holds with equality. The CAPM implies such a situation, so that the slope of the capital market line, $S_e \equiv \frac{E[R_m]-R_f}{\sigma_{R_m}}$, equals $\sigma_{m_{01}} R_f$.

# Ex: Bounds with Power Utility

- If $U(C) = C^{\gamma}/\gamma$ so $m_{01} \equiv \delta (C_1/C_0)^{\gamma-1} = \delta e^{(\gamma-1)\ln(C_1/C_0)}$ and $C_1/C_0$ is lognormal with parameters $\mu_c$ and $\sigma_c$, then

$$
\begin{aligned}
\frac{\sigma_{m_{01}}}{E[m_{01}]} &= \frac{\sqrt{Var\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]}}{E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]} \\
&= \frac{\sqrt{E\left[e^{2(\gamma-1)\ln(C_1/C_0)}\right] - E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]^2}}{E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]} \\
&= \sqrt{E\left[e^{2(\gamma-1)\ln(C_1/C_0)}\right]/E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]^2 - 1} \\
&= \sqrt{e^{2(\gamma-1)\mu_c + 2(\gamma-1)^2\sigma_c^2}/e^{2(\gamma-1)\mu_c + (\gamma-1)^2\sigma_c^2} - 1} = \sqrt{e^{(\gamma-1)^2\sigma_c^2} - 1} \\
&\approx \pm(\gamma-1)\sigma_c = (1-\gamma)\sigma_c \qquad (32)
\end{aligned}
$$

The fourth line evaluates expectations assuming $C_1$ log-normality, $E(X) = e^{\mu + \frac{1}{2}\sigma^2}$.The fifth line takes a two-term approximation of the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$, which is reasonable for small positive $x$. The $(+)$ solution is negative for $\gamma < 1$.

# Ex: Bounds with Power Utility

- Hence, with power utility and lognormal consumption:

$$\left| \frac{E\left[R_i\right] - R_f}{\sigma_{R_i}} \right| \leq (1 - \gamma)\, \sigma_c \qquad (33)$$

- For the S&P500 over the last 75 years, $E\left[R_i\right] - R_f = 8.3\%$ and $\sigma_{R_i} = .17$, implying a Sharpe ratio of $\frac{E[R_i]-R_f}{\sigma_{R_i}} = 0.49$.
- U.S. per capita consumption data implies estimates of $\sigma_c$ between 0.01 and 0.0386.
- Assuming (33) holds with equality for the S&P500, $\gamma = 1 - \left(\frac{E[R_i]-R_f}{\sigma_{R_i}}\right)/\sigma_c$ is between -11.7 and -48.
- Other empirical estimates of $\gamma$ are -1 to -5. The inconsistency of theory and empirical evidence is what Mehra and Prescott (1985) termed the *equity premium puzzle*.

# Ex. Bounds on $R_f$

- Even if high risk aversion is accepted, it implies an unreasonable value for the risk-free return, $R_f$. Note that

$$\frac{1}{R_f} = E\left[m_{01}\right] \tag{34}$$
$$= \delta E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]$$
$$= \delta e^{(\gamma-1)\mu_c+\frac{1}{2}(\gamma-1)^2\sigma_c^2}$$

and therefore

$$\ln\left(R_f\right) = -\ln\left(\delta\right) + \left(1-\gamma\right)\mu_c - \frac{1}{2}\left(1-\gamma\right)^2\sigma_c^2 \tag{35}$$

- If we set $\delta = 0.99$, and $\mu_c = 0.018$, the historical average real growth of U.S. per capita consumption, then with $\gamma = -11$ and $\sigma_c = 0.036$ we obtain:

# Ex. Bounds on $R_f$ cont'd

$$\begin{aligned} \ln(R_f) &= -\ln(\delta) + (1-\gamma)\mu_c - \frac{1}{2}(1-\gamma)^2\sigma_c^2 \\ &= 0.01 + 0.216 - 0.093 = 0.133 \end{aligned} \tag{36}$$

which is a real risk-free interest rate of 13.3 percent.

- Since short-term real interest rates have averaged about 1 percent in the U.S., we end up with a *risk-free rate puzzle*: the high $\gamma$ results in an unreasonable $R_f$.
- So a SDF derived from the marginal utility of consumption doesn't fit the data. However, we can derive a SDF of the form $P_i = E_0[m_{01}X_i]$ using another approach.

# Complete Markets Assumptions

- An alternative SDF derivation is based on the assumptions of a complete market and the absence of arbitrage.
- Suppose that an individual can freely trade in $n$ assets and assume that there is a finite number, $k$, of end-of-period states of nature, with state $s$ having probability $\pi_s$.
- Let $X_{si}$ be the cashflow returned by one share (unit) of asset $i$ in state $s$. Asset $i$'s cashflows can be written as:

$$X_i = \begin{bmatrix} X_{1i} \\ \vdots \\ X_{ki} \end{bmatrix} \tag{37}$$

# Complete Markets Assumptions cont'd

- Thus, the per-share cashflows of the universe of all assets can be represented by the $k \times n$ matrix

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{k1} & \cdots & X_{kn} \end{bmatrix} \quad (38)$$

- We will assume that $n = k$ and that $X$ is of full rank, implying that the $n$ assets *span* the $k$ states of nature and the market is *complete*.
- An implication is that an individual can purchase amounts of the $k$ assets that return target levels of end-of-period wealth in each of the states.

# Complete Markets Assumptions cont'd

- To show this complete markets result, let $W$ be an arbitrary $k \times 1$ vector of end-of-period levels of wealth:

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_k \end{bmatrix} \tag{39}$$

  where $W_s$ is the level of wealth in state $s$.

- To obtain $W$, at the initial date the individual purchases shares in the $k$ assets. Let the vector $N = [N_1 \ \ldots N_k]'$ be the number of shares purchased of each of the $k$ assets. Hence, $N$ must satisfy

$$XN = W \tag{40}$$

# Complete Markets Assumptions cont'd

- Since $X$ is nonsingular, its inverse exists so that

$$N = X^{-1} W \tag{41}$$

is the unique solution.

- Denoting $P = [P_1 \ \ldots \ P_k]'$ as the $k \times 1$ vector of beginning-of-period, per-share prices of the $k$ assets, then the initial wealth required to produce the target level of wealth given in (39) is $P'N$.
- The absence of arbitrage implies that the price of a new, redundant security or contingent claim that pays $W$ is determined from the prices of the original $k$ securities, and this claim's price must be $P'N$.

# Arbitrage and State Prices

- Consider the case of a *primitive*, *elementary*, or *Arrow-Debreu* security which has a payoff of 1 in state $s$ and 0 in all other states:

$$e_s = \begin{bmatrix} W_1 \\ \vdots \\ W_{s-1} \\ W_s \\ W_{s+1} \\ \vdots \\ W_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{42}$$

# Arbitrage and State Prices

- Then $p_s$, the price of elementary security $s$, is

$$p_s = P'X^{-1}e_s, \quad s = 1, ..., k \tag{43}$$

  so a unique set of state prices exists in a complete market.
- These elementary state prices should each be positive, since wealth received in any state will have positive value when individuals are nonsatiated. Hence (43) and $p_s > 0 \ \forall s$ restrict the payoffs, $X$, and the prices, $P$, of the original $k$ securities.
- Note that the portfolio composed of the sum of all elementary securities gives a cashflow of 1 unit with certainty and determines the risk-free return, $R_f$:

# Arbitrage and State Prices cont'd

$$\sum_{s=1}^{k} p_s = \frac{1}{R_f} \tag{44}$$

- For a general multicashflow asset, $a$, whose cashflow in state $s$ is $X_{sa}$, absence of arbitrage ensures its price, $P_a$, is

$$P_a = \sum_{s=1}^{k} p_s X_{sa} \tag{45}$$

- Consider the connection to state probabilities, $\pi_s$, by defining $m_s \equiv p_s/\pi_s$. Since $p_s > 0\ \forall s$, then $m_s > 0\ \forall s$ when $\pi_s > 0$.

# Arbitrage and State Prices cont'd

- Then equation (45) can be written as

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s \frac{p_s}{\pi_s} X_{sa} \\ &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E\left[m X_a\right] \end{aligned} \tag{46}$$

  where $m$ denotes a stochastic discount factor whose expected value is $E\left[m\right] = \sum_{s=1}^{k} \pi_s m_s = \sum_{s=1}^{k} p_s = 1/R_f$, and $X_a$ is the random cashflow of the multicashflow asset $a$.
- In terms of the consumption-based model, $m_s = \delta U'\left(C_{1s}\right)/U'\left(C_0\right)$ where $C_{1s}$ is consumption at date 1 in state $s$, $p_s$ is greater when $C_{1s}$ is low.

# Risk-Neutral Probabilities

- Define $\widehat{\pi}_s \equiv p_s R_f$. Then

$$\begin{aligned} P_a &= \sum_{s=1}^{k} p_s X_{sa} \\ &= \frac{1}{R_f} \sum_{s=1}^{k} p_s R_f X_{sa} \\ &= \frac{1}{R_f} \sum_{s=1}^{k} \widehat{\pi}_s X_{sa} \end{aligned} \tag{47}$$

- Now $\widehat{\pi}_s$, $s = 1, ..., k$, have the characteristics of probabilities because they are positive, $\widehat{\pi}_s = p_s / \sum_{s=1}^{k} p_s > 0$, and they sum to 1, $\sum_{s=1}^{k} \widehat{\pi}_s = R_f \sum_{s=1}^{k} p_s = R_f / R_f = 1$.

# Risk-Neutral Probabilities cont'd

- Using this insight, equation (47) can be written

$$
\begin{aligned}
P_a &= \frac{1}{R_f} \sum\nolimits_{s=1}^{k} \widehat{\pi}_s X_{sa} \\
&= \frac{1}{R_f} \widehat{E}\left[X_a\right] \qquad (48)
\end{aligned}
$$

  where $\widehat{E}\left[\cdot\right]$ denotes the expectation operator using the "pseudo" probabilities $\widehat{\pi}_s$ rather than the true probabilities $\pi_s$.
- Since the expectation in (48) is discounted by the risk-free return, we can recognize $\widehat{E}\left[X_a\right]$ as the certainty equivalent expectation of the cashflow $X_a$.

# Risk-Neutral Probabilities cont'd

- Since $m_s \equiv p_s/\pi_s$ and $R_f = 1/E[m]$, $\widehat{\pi}_s$ can be written as

$$\begin{aligned} \widehat{\pi}_s &= R_f p_s = R_f m_s \pi_s \\ &= \frac{m_s}{E[m]} \pi_s \end{aligned} \qquad (49)$$

- In states where the SDF $m_s$ is greater than its average, $E[m]$, the pseudo probability exceeds the true probability.
- Note if $m_s = \frac{1}{R_f} = E[m]$ then $P_a = E[mX_a] = E[X_a]/R_f$ so the price equals the expected payoff discounted at the risk-free rate, as if investors were risk-neutral.

# Risk-Neutral Probabilities cont'd

- Hence, $\widehat{\pi}_s$ is referred to as the *risk-neutral* probability.
- $\widehat{E}[\cdot]$, also often denoted as $E^Q[\cdot]$, is referred to as the risk-neutral expectations operator.
- In comparison, the true probabilities, $\pi_s$, are frequently called the *physical,* or *statistical*, probabilities.

# Two-State Example

- Suppose $k = 2$ with state 1 being an economic expansion and state 2 being an economic contraction, and $\pi_1 = \pi_2 = \frac{1}{2}$.
- A default-free bond pays 1 in both states, but a default-risky bond pays 1 in state 1 and $\frac{1}{2}$ in state 2. Thus

$$X = \begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \tag{50}$$

- The price of the default-free bond is $\frac{1}{R_f} = \frac{1}{1+r_f} = 1$, implying $r_f = 0$, and the price of the default-risky bond is 0.7, so that

$$P = \begin{bmatrix} 1 \\ 0.7 \end{bmatrix} \tag{51}$$

# Example: State Price Valuation

- The elementary security prices are

$$p_1 = P'X^{-1}e_1 = \begin{bmatrix} 1 & 0.7 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.4$$

$$p_2 = P'X^{-1}e_2 = \begin{bmatrix} 1 & 0.7 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.6$$

- If Stock $a$'s cashflows are 130 in state 1 and 80 in state 2:

$$\begin{aligned} P_a &= \textstyle\sum_{s=1}^{2} p_s X_{sa} = \\ &= 0.4 \times 130 + 0.6 \times 80 = 52 + 48 = 100 \end{aligned} \tag{52}$$

so that the stock's expected return is

$$E\left[\widetilde{R}_a\right] = E\left[\widetilde{X}_a\right]/P_a = \left(\tfrac{1}{2} \times 130 + \tfrac{1}{2} \times 80\right)/100 = 1.05.$$

# Example: SDF Valuation

- Now $m_1 = \frac{p_1}{\pi_1} = \frac{0.4}{\frac{1}{2}} = 0.8$ and $m_2 = \frac{p_2}{\pi_2} = \frac{0.6}{\frac{1}{2}} = 1.2$.

- Therefore, using the SDF valuation of Stock *a* leads to

$$
\begin{aligned}
P_a &= E\left[m\, X_a\right] \\
&= \textstyle\sum_{s=1}^{2} \pi_s m_s X_{sa} \\
&= \frac{1}{2} \times 0.8 \times 130 + \frac{1}{2} \times 1.2 \times 80 \\
&= 52 + 48 = 100
\end{aligned}
$$

# Example: Risk-Neutral Valuation

- Finally, note that $\widehat{\pi}_1 \equiv p_1 R_f = 0.4 \times 1$ and $\widehat{\pi}_2 \equiv p_2 R_f = 0.6 \times 1$.
- Therefore, using risk-neutral valuation of Stock $a$ leads to

$$\begin{aligned} P_a &= \frac{1}{R_f} \widehat{E}\left[X_a\right] \qquad (53) \\ &= \frac{1}{R_f} \sum_{s=1}^{2} \widehat{\pi}_s X_{sa} \\ &= \frac{1}{1}\left(0.4 \times 130 + 0.6 \times 80\right) = 100 \end{aligned}$$

- So all three methods lead to the same valuation because they are mathematically equivalent.

# State Pricing Extensions

- This complete markets pricing, also known as *State Preference Theory,* can be generalized to an infinite number of states and elementary securities.
- Suppose states are indexed by all possible points on the real line between 0 and 1; that is, the state $s \in (0, 1)$.
- Also let $p(s)$ be the price (density) of a primitive security that pays 1 unit in state $s$, 0 otherwise.

# State Pricing Extensions cont'd

- Further, define $X_a(s)$ as the cashflow paid by security $a$ in state $s$.
- Then, analogous to (44),

$$\int_0^1 p(s)\,ds = \frac{1}{R_f} \tag{54}$$

and the price of security $a$ is

$$P_a = \int_0^1 p(s)\,X_a(s)\,ds \tag{55}$$

# State Pricing Extensions cont'd

- In *Time State Preference Theory*, assets pay cashflows at different dates in the future and markets are complete.
- For example, an asset may pay cashflows at both date 1 and date 2 in the future: let $s_1$ be a state at date 1 and let $s_2$ be a state at date 2. States at date 2 can depend on which states were reached at date 1.
- Suppose there are two events at each date, economic recession ($r$) or economic expansion (boom) ($b$). Then define $s_1 \in \{r_1, b_1\}$ and $s_2 \in \{r_1 r_2, r_1 b_2, b_1 r_2, b_1 b_2\}$.
- By assigning suitable probabilities and primitive security state prices for assets that pay cashflows of 1 unit in each of these six states, we can sum (or integrate) over both time and states at a given date to obtain prices of complex securities.

# Summary

- An optimal portfolio is one where assets' expected marginal utility-weighted returns are equalized, and the individual's optimal savings trades off expected marginal utility of current and future consumption.
- Assets can be priced using a SDF that is the marginal rate of substitution between current and future consumption.
- A SDF can also be derived based on assumptions of market completeness and no arbitrage.
- A risk-neutral pricing formula transforms physical probabilities to account for risk.