CAPM, Arbitrage, and Linear Factor Models

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Introduction

- We now assume all investors actually choose mean-variance efficient portfolios.
- By equating these investors' aggregate asset demands to aggregate asset supply, an equilibrium single risk factor pricing model (CAPM) can be derived.
- Relaxing CAPM assumptions may allow for *multiple* risk factors.
- Arbitrage arguments can be used to derive a multifactor pricing model (APT)
- Multifactor models are very popular in empirical asset pricing

Review of Mean-Variance Portfolio Choice

- Recall that for n risky assets and a risk-free asset, the optimal portfolio weights for the n risky assets are

$$\omega^* = \lambda V^{-1} \left(\bar{R} - R_f e \right) \tag{1}$$

where $\lambda \equiv \dfrac{\overline{R}_p - R_f}{\left(\bar{R} - R_f e \right)' V^{-1} \left(\bar{R} - R_f e \right)}$.

- The amount invested in the risk-free asset is then $1 - e'\omega^*$.
- $\overline{R}_p$ is determined by where the particular investor's indifference curve is tangent to the efficient frontier.
- All investors, no matter what their risk aversion, choose the risky assets in the same *relative* proportions.

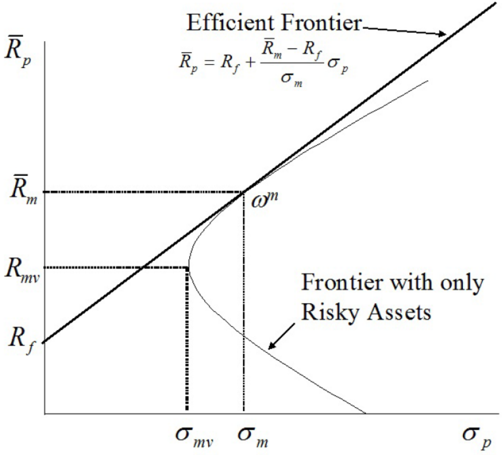
Tangency Portfolio

- Also recall that the efficient frontier is linear in σ_p and $\overline{R}_p$:

$$\overline{R}_p = R_f + \left(\left(\bar{R} - R_f e\right)' V^{-1} \left(\bar{R} - R_f e\right) \right)^{\frac{1}{2}} \sigma_p \qquad (2)$$

- This frontier is tangent to the "risky asset only" frontier, where the following graph denotes this tangency portfolio as ω^m located at point σ_m, $\overline{R}_m$.

Graph of Efficient Frontier



The Tangency Portfolio

- Note that the tangency portfolio satisfies $e'\omega^m = 1$. Thus

$$e'\lambda V^{-1}\left(\bar{R} - R_f e\right) = 1 \tag{3}$$

or

$$\lambda = m \equiv \left[\left(\overline{R} - R_f e\right)' V^{-1} e\right]^{-1} \tag{4}$$

so that

$$\omega^m = mV^{-1}(\bar{R} - R_f e) \tag{5}$$

Asset Covariances with the Tangency Portfolio

- Now define σ_M as the $n \times 1$ vector of covariances of the tangency portfolio with each of the n risky assets. It equals

$$\sigma_M = V\omega^m = m(\bar{R} - R_f e) \tag{6}$$

- By pre-multiplying equation (6) by $\omega^{m\prime}$, we also obtain the variance of the tangency portfolio:

$$\begin{aligned} \sigma_m^2 &= \omega^{m\prime} V \omega^m = \omega^{m\prime}\sigma_M = m\omega^{m\prime}(\bar{R} - R_f e) \\ &= m(\bar{R}_m - R_f) \end{aligned} \tag{7}$$

where $\bar{R}_m \equiv \omega^{m\prime}\bar{R}$ is the expected return on the tangency portfolio.

Expected Excess Returns

- Rearranging (6) and substituting in for $\frac{1}{m} = \frac{1}{\sigma_m^2}(\overline{R}_m - R_f)$ from (7), we have

$$(\bar{R} - R_f e) = \frac{1}{m}\sigma_M = \frac{\sigma_M}{\sigma_m^2}(\bar{R}_m - R_f) = \beta(\bar{R}_m - R_f) \qquad (8)$$

where $\beta \equiv \frac{\sigma_M}{\sigma_m^2}$ is the $n \times 1$ vector whose i^{th} element is $\frac{Cov(\tilde{R}_m, \tilde{R}_i)}{Var(\tilde{R}_m)}$.

- Equation (8) links the excess expected return on the tangency portfolio, $(\bar{R}_m - R_f)$, to the excess expected returns on the individual risky assets, $(\bar{R} - R_f e)$.

CAPM

- The Capital Asset Pricing Model is completed by noting that the tangency portfolio, ω^m, chosen by all investors must be the equilibrium market portfolio.
- Hence, $\overline{R}_m$ and σ_m^2 are the mean and variance of the market portfolio returns and σ_M is its covariance with the individual assets.
- Aggregate supply can be modeled in different ways (endowment economy, production economy), but in equilibrium it will equal aggregate demands for the risky assets in proportions given by ω^m.
- Also $R_f < R_{mv}$ for assets to be held in positive amounts ($\omega_i^m > 0$).

CAPM: Realized Returns

- Define asset i's and the market's realized returns as $\tilde{R}_i = \bar{R}_i + \tilde{\nu}_i$ and $\tilde{R}_m = \bar{R}_m + \tilde{\nu}_m$ where $\tilde{\nu}_i$ and $\tilde{\nu}_m$ are the unexpected components. Substitute these into (8):

$$\begin{aligned} \tilde{R}_i &= R_f + \beta_i(\tilde{R}_m - \tilde{\nu}_m - R_f) + \tilde{\nu}_i \qquad (9) \\ &= R_f + \beta_i(\tilde{R}_m - R_f) + \tilde{\nu}_i - \beta_i \tilde{\nu}_m \\ &= R_f + \beta_i(\tilde{R}_m - R_f) + \widetilde{\varepsilon}_i \end{aligned}$$

where $\widetilde{\varepsilon}_i \equiv \tilde{\nu}_i - \beta_i \tilde{\nu}_m$. Note that

$$\begin{aligned} Cov(\tilde{R}_m, \widetilde{\varepsilon}_i) &= Cov(\tilde{R}_m, \tilde{\nu}_i) - \beta_i Cov(\tilde{R}_m, \tilde{\nu}_m) \qquad (10) \\ &= Cov(\tilde{R}_m, \tilde{R}_i) - \frac{Cov(\tilde{R}_m, \tilde{R}_i)}{Var(\tilde{R}_m)} Cov(\tilde{R}_m, \tilde{R}_m) \\ &= Cov(\tilde{R}_m, \tilde{R}_i) - Cov(\tilde{R}_m, \tilde{R}_i) = 0 \end{aligned}$$

Idiosyncratic Risk

- Since $Cov(\tilde{R}_m, \widetilde{\varepsilon}_i) = 0$, from equation (9) we see that the total variance of risky asset i, σ_i^2, equals:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2 \tag{11}$$

- Another implication of $Cov(\tilde{R}_m, \widetilde{\varepsilon}_i) = 0$ is that equation (9) represents a regression equation.
- The orthogonal, mean-zero residual, $\widetilde{\varepsilon}_i$, is referred to as idiosyncratic, unsystematic, or diversifiable risk.
- Since this portion of the asset's risk can be eliminated by the individual who invests optimally, there is no "price" or "risk premium" attached to it.

What Risk is Priced?

- To make clear what risk is priced, denote $\sigma_{Mi} = Cov(\tilde{R}_m, \widetilde{R}_i)$, which is the i^{th} element of σ_M. Also let ρ_{im} be the correlation between $\widetilde{R}_i$ and $\tilde{R}_m$.
- Then equation (8) can be rewritten as

$$\bar{R}_i - R_f = \frac{\sigma_{Mi}}{\sigma_m^2}(\bar{R}_m - R_f) = \frac{\sigma_{Mi}}{\sigma_m}\frac{(\bar{R}_m - R_f)}{\sigma_m} = \rho_{mi}\sigma_i \frac{(\bar{R}_m - R_f)}{\sigma_m} = \rho_{mi}\sigma_i S_e \qquad (12)$$

where $S_e \equiv \frac{(\bar{R}_m - R_f)}{\sigma_m}$ is the equilibrium excess return on the market portfolio per unit of market risk.

What Risk is Priced? cont'd

- $S_e \equiv \frac{(\bar{R}_m - R_f)}{\sigma_m}$ is known as the market Sharpe ratio.
- It represents "the market price" of systematic or nondiversifiable risk, and is also referred to as the slope of the *capital market line*, where the capital market line is the efficient frontier that connects the points R_f and $\left(\overline{R}_{m,}\sigma_m\right)$.
- Now define ω_i^m as the weight of asset i in the market portfolio and V_i as the i^{th} row of V. Then

$$\frac{\partial \sigma_m}{\partial \omega_i^m} = \frac{1}{2\sigma_m}\frac{\partial \sigma_m^2}{\partial \omega_i^m} = \frac{1}{2\sigma_m}\frac{\partial \omega^m V \omega^m}{\partial \omega_i^m} = \frac{1}{2\sigma_m} 2V_i \omega^m = \frac{1}{\sigma_m}\sum_{j=1}^{n} \omega_j^m \sigma_{ij} \tag{13}$$

where σ_{ij} is the i, j^{th} element of V.

What Risk is Priced? cont'd

- Since $\tilde{R}_m = \sum_{j=1}^{n} \omega_j^m \tilde{R}_j$, then $Cov(\tilde{R}_i, \tilde{R}_m) = \sum_{j=1}^{n} \omega_j^m \sigma_{ij}$. Hence, (13) is

$$\frac{\partial \sigma_m}{\partial \omega_i^m} = \frac{1}{\sigma_m} Cov(\tilde{R}_i, \tilde{R}_m) = \rho_{im} \sigma_i \tag{14}$$

- Thus, $\rho_{im}\sigma_i$ is the marginal increase in "market risk," σ_m, from a marginal increase of asset i in the market portfolio. Thus $\rho_{im}\sigma_i$ is the *quantity* of asset i's systematic risk.
- We saw in (12) that $\rho_{im}\sigma_i$ multiplied by the price of systematic risk, S_e, determines an asset's required risk premium.

Zero-Beta CAPM (Black, 1972)

- Does a CAPM hold when there is no riskless asset?
- Suppose the economy has I total investors with investor i having a *proportion* W_i of the economy's total initial wealth and choosing an efficient frontier portfolio $\omega^i = a + b\bar{R}_{ip}$, where $\bar{R}_{ip}$ reflects investor i's risk aversion.
- Then the risky asset weights of the market portfolio are

$$\begin{aligned} \omega^m &= \sum_{i=1}^{I} W_i \omega^i = \sum_{i=1}^{I} W_i \left(a + b\bar{R}_{ip}\right) \\ &= a \sum_{i=1}^{I} W_i + b \sum_{i=1}^{I} W_i \bar{R}_{ip} = a + b\bar{R}_m \end{aligned} \tag{15}$$

where $\bar{R}_m \equiv \sum_{i=1}^{I} W_i \bar{R}_{ip}$.

Zero-Beta CAPM cont'd

- Equation (15) shows that the aggregate market portfolio, ω^m, is a frontier portfolio and its expected return, $\overline{R}_m$, is a weighted average of the expected returns of the individual investors' portfolios.
- Consider the covariance between the market portfolio and an arbitrary risky portfolio with weights ω^0, random return of $\widetilde{R}_{0p}$, and mean return of $\overline{R}_{0p}$:

$$Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right) = \omega^{m\prime} V \omega^0 = \left(a + b\bar{R}_m\right)' V\omega^0 \qquad (16)$$

$$= \left(\frac{\varsigma V^{-1} e - \alpha V^{-1} \bar{R}}{\varsigma\delta - \alpha^2} + \frac{\delta V^{-1}\bar{R} - \alpha V^{-1} e}{\varsigma\delta - \alpha^2}\bar{R}_m\right)' V\omega^0$$

Zero-Beta CAPM cont'd

$$
\begin{aligned}
&= \frac{\varsigma e' V^{-1} V \omega^0 - \alpha \bar{R}' V^{-1} V \omega^0}{\varsigma\delta - \alpha^2} \\
&\quad + \frac{\delta \bar{R}_m \bar{R}' V^{-1} V \omega^0 - \alpha \bar{R}_m e' V^{-1} V \omega^0}{\varsigma\delta - \alpha^2} \\
&= \frac{\varsigma - \alpha \overline{R}_{0p} + \delta \bar{R}_m \overline{R}_{0p} - \alpha \bar{R}_m}{\varsigma\delta - \alpha^2}
\end{aligned}
$$

Rearranging (16) gives

$$
\overline{R}_{0p} = \frac{\alpha \bar{R}_m - \varsigma}{\delta \bar{R}_m - \alpha} + Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right) \frac{\varsigma\delta - \alpha^2}{\delta \bar{R}_m - \alpha} \tag{17}
$$

Recall from (2.32) that $\sigma_m^2 = \frac{1}{\delta} + \frac{\delta\left(\overline{R}_m - \frac{\alpha}{\delta}\right)^2}{\varsigma\delta - \alpha^2}$

Zero-Beta CAPM cont'd

- Multiply and divide the second term of (17) by σ_m^2, and add and subtract $\frac{\alpha^2}{\delta}$ from the top and factor out δ from the bottom of the first term to obtain:

$$\begin{aligned}\overline{R}_{0p} &= \frac{\alpha}{\delta} - \frac{\varsigma\delta - \alpha^2}{\delta^2\left(\overline{R}_m - \frac{\alpha}{\delta}\right)} + \frac{Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right)}{\sigma_m^2}\left(\frac{1}{\delta} + \frac{\delta\left(\overline{R}_m - \frac{\alpha}{\delta}\right)^2}{\varsigma\delta - \alpha^2}\right)\frac{\varsigma\delta - \alpha^2}{\delta\bar{R}_m - \alpha} \\ &= \frac{\alpha}{\delta} - \frac{\varsigma\delta - \alpha^2}{\delta^2\left(\overline{R}_m - \frac{\alpha}{\delta}\right)} + \frac{Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right)}{\sigma_m^2}\left(\overline{R}_m - \frac{\alpha}{\delta} + \frac{\varsigma\delta - \alpha^2}{\delta^2\left(\overline{R}_m - \frac{\alpha}{\delta}\right)}\right)\end{aligned} \tag{18}$$

- From (2.39), the first two terms of (18) equal the expected return on the portfolio that has zero covariance with the market portfolio, call it $\overline{R}_{zm}$. Thus, equation (18) can be written as

Zero-Beta CAPM cont'd

$$\begin{aligned}\overline{R}_{0p} &= \overline{R}_{zm} + \frac{Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right)}{\sigma_m^2}\left(\overline{R}_m - \overline{R}_{zm}\right) \\ &= \overline{R}_{zm} + \beta_0\left(\overline{R}_m - \overline{R}_{zm}\right)\end{aligned} \tag{19}$$

- Since ω^0 is any risky-asset portfolio, including a single asset portfolio, (19) is identical to the previous CAPM result (12) except that $\overline{R}_{zm}$ replaces R_f.
- Thus, a single factor CAPM continues to exist when there is no riskless asset.
- Stephen Ross (1976) derived a similar multifactor relationship, not based on investor preferences but rather the principle of arbitrage.

Arbitrage

- If a portfolio with zero net investment cost can produce gains, but never losses, it is an arbitrage portfolio.
- In efficient markets, arbitrage should be temporary: exploitation by investors moves prices to eliminate it.
- If equilibrium prices do not permit arbitrage, then the law of one price holds:

If different assets produce the same future payoffs, then the current prices of these assets must be the same.

- Not all markets are always in equilibrium: in some cases arbitrage trades may not be possible.

Example: Covered Interest Parity

- *Covered interest parity* links spot and forward foreign exchange markets to foreign and domestic money markets.
- Let $F_{0\tau}$ be the current date 0 forward price for exchanging one unit of a foreign currency τ periods in the future, e.g. the dollar price to be paid τ periods in the future for delivery of one unit of foreign currency τ periods in the future.
- Let S_0 be the spot price of foreign exchange, that is, the current date 0 dollar price of one unit of foreign currency to be delivered immediately.
- Also let R_f be the per-period risk-free (money market) return for borrowing or lending in dollars over the period 0 to τ, and denote as R_f^* the per-period risk-free return for borrowing or lending in the foreign currency over the period 0 to τ.

Covered Interest Parity cont'd

- Now let's construct a zero net cost portfolio.

1. Sell forward one unit of foreign currency at price $F_{0\tau}$ (0 cost).
2. Purchase the present value of one unit of foreign currency, $1/R_f^{*\tau}$ and invest in a foreign bond at R_f^*. ($S_0/R_f^{*\tau}$ cost).
3. Borrow $S_0/R_f^{*\tau}$ dollars at the per-period return R_f ($-S_0/R_f^{*\tau}$ cost)

- At date τ, note that the foreign currency investment yields $R_f^{*\tau}/R_f^{*\tau} = 1$ unit of the foreign currency, which covers the short position in the forward foreign exchange contract.
- For delivering this foreign currency, we receive $F_{0\tau}$ dollars but we also owe a sum of $R_f^\tau S_0/R_f^{*\tau}$ due to our dollar borrowing. Thus, our net proceeds at date τ are

$$F_{0\tau} - R_f^\tau S_0/R_f^{*\tau} \tag{20}$$

Covered Interest Parity cont'd

- These proceeds are nonrandom; they depend only on prices and riskless rates quoted at date 0.
- If this amount is positive (*negative*), then we should buy (*sell*) this portfolio as it represents an arbitrage.
- Thus, in the absence of arbitrage, it must be that

$$F_{0\tau} = S_0 R_f^{\tau} / R_f^{*\tau} \qquad (21)$$

  which is the *covered interest parity* condition: given S_0, R_f^* and R_f, the forward rate is pinned down.
- Thus, when applicable, pricing assets or contracts by ruling out arbitrage is attractive in that assumptions regarding investor preferences or beliefs are not required.

Example: Arbitrage and CAPM

- Suppose a single source of (market) risk determines all risky-asset returns according to the linear relationship

$$\widetilde{R}_i = a_i + b_i \tilde{f} \tag{22}$$

where $\widetilde{R}_i$ is the i^{th} asset's return and $\tilde{f}$ is a single risk factor generating all asset returns, where $E[\tilde{f}] = 0$ is assumed.

- a_i is asset i's expected return, that is, $E[\widetilde{R}_i] = a_i$.
- b_i is the sensitivity of asset i to the risk factor (asset i's beta coefficient).
- There is also a riskfree asset returning R_f.
- Now construct a portfolio of two assets, with a proportion ω invested in asset i and $(1 - \omega)$ invested in asset j.

Arbitrage and CAPM cont'd

- This portfolio's return is given by

$$\begin{aligned} \widetilde{R}_p &= \omega a_i + (1-\omega) a_j + \omega b_i \tilde{f} + (1-\omega) b_j \tilde{f} \\ &= \omega (a_i - a_j) + a_j + [\omega (b_i - b_j) + b_j] \tilde{f} \end{aligned} \tag{23}$$

- If the portfolio weights are chosen such that

$$\omega^* = \frac{b_j}{b_j - b_i} \tag{24}$$

then the random component of the portfolio's return is eliminated: R_p is risk-free.

- The absence of arbitrage requires $R_p = R_f$, so that

$$R_p = \omega^* (a_i - a_j) + a_j = R_f \tag{25}$$

Arbitrage and CAPM cont'd

- Rearranging this condition

$$\frac{b_j(a_i - a_j)}{b_j - b_i} + a_j = R_f$$

$$\frac{b_j a_i - b_i a_j}{b_j - b_i} = R_f$$

which can also be written as

$$\frac{a_i - R_f}{b_i} = \frac{a_j - R_f}{b_j} \equiv \lambda \tag{26}$$

which states that expected excess returns, per unit of risk, must be equal for all assets. λ is defined as this risk premium per unit of factor risk.

Arbitrage and CAPM cont'd

- Equation (26) is a fundamental relationship, and similar law-of-one-price conditions hold for virtually all asset pricing models.
- For example, we can rewrite the CAPM equation (12) as

$$\frac{\bar{R}_i - R_f}{\rho_{im}\sigma_i} = \frac{(\bar{R}_m - R_f)}{\sigma_m} \equiv S_e \tag{27}$$

so that the ratio of an asset's expected return premium, $\bar{R}_i - R_f$, to its quantity of market risk, $\rho_{im}\sigma_i$, is the same for all assets and equals the slope of the capital market line, S_e.

- What can arbitrage arguments tell us about more general models of risk factor pricing?

Linear Factor Models

- The CAPM assumption that all assets can be held by all individual investors is clearly an oversimplification.
- In addition to the risk from returns on a global portfolio of marketable assets, individuals are likely to face multiple sources of nondiversifiable risks. This is a motivation for the multifactor Arbitrage Pricing Theory (APT) model.
- APT does not make assumptions about investor preferences but uses arbitrage pricing to restrict an asset's risk premium.
- Assume that there are k risk factors and n assets in the economy, where $n > k$.
- Let b_{iz} be the sensitivity of the i^{th} asset to the z^{th} risk factor, where $\tilde{f}_z$ is the random realization of risk factor z.

Linear Factor Models cont'd

- Let $\widetilde{\varepsilon}_i$ be idiosyncratic risk specific to asset i, which by definition is independent of the k risk factors, $\tilde{f}_1,...,\tilde{f}_k$, and the specific risk of any other asset j, $\widetilde{\varepsilon}_j$.
- For a_i, the expected return on asset i, the linear return-generating process is assumed to be

$$\widetilde{R}_i = a_i + \sum_{z=1}^{k} b_{iz} \tilde{f}_z + \widetilde{\varepsilon}_i \tag{28}$$

where $E\left[\widetilde{\varepsilon}_i\right] = E\left[\tilde{f}_z\right] = E\left[\widetilde{\varepsilon}_i \tilde{f}_z\right] = 0$ and $E\left[\widetilde{\varepsilon}_i \widetilde{\varepsilon}_j\right] = 0 \ \forall \ i \neq j$.

- The risk factors are transformed to be mutually independent and normalized to unit variance, $E\left[\tilde{f}_z^2\right] = 1$.

Linear Factor Models cont'd

- Finally, the idiosyncratic variance is assumed finite:

$$E\left[\tilde{\varepsilon}_i^2\right] \equiv s_i^2 < S^2 < \infty \tag{29}$$

- Note $Cov\left(\widetilde{R}_i, \tilde{f}_z\right) = Cov\left(b_{iz}\tilde{f}_z, \tilde{f}_z\right) = b_{iz}Cov\left(\tilde{f}_z, \tilde{f}_z\right) = b_{iz}$.
- In the previous example there was only systematic risk, which we could eliminate with a hedge portfolio. Now each asset's return contains an idiosyncratic risk component.
- We will use the notion of *asymptotic arbitrage* to argue that assets' expected returns will be "close" to the relationship that would result if they had no idiosyncratic risk, since it is diversifiable with a large number of assets.

Arbitrage Pricing Theory

- Consider a portfolio of n assets where σ_{ij} is the covariance between the returns on assets i and j and the portfolio's investment *amounts* are $W^n \equiv [W_1^n \ W_2^n \dots W_n^n]'$.
- Consider a sequence of these portfolios where the number of assets in the economy is increasing, $n = 2, 3, \ldots$.
- Definition: An *asymptotic arbitrage* exists if:
  (A) The portfolio requires zero net investment:

$$\sum_{i=1}^{n} W_i^n = 0$$

  (B) The portfolio return becomes certain as n gets large:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} W_i^n W_j^n \sigma_{ij} = 0$$

Arbitrage Pricing Theory cont'd

(C) The portfolio's expected return is always bounded above zero

$$\sum_{i=1}^{n} W_i^n a_i \geq \delta > 0$$

Theorem: If no asymptotic arbitrage opportunities exist, then the expected return of asset i, $i = 1, ..., n$, is described by the following linear relation:

$$a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + \nu_i \qquad (*)$$

where λ_0 is a constant, λ_z is the risk premium for risk factor $\widetilde{f}_z$, $z = 1, ..., k$, and the expected return deviations, ν_i, satisfy

Arbitrage Pricing Theory cont'd

$$\sum_{i=1}^{n} \nu_i = 0 \qquad (i)$$

$$\sum_{i=1}^{n} b_{iz}\nu_i = 0, \quad z = 1, ..., k \qquad (ii)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \nu_i^2 = 0 \qquad (iii)$$

- Note that (iii) says that the average squared error (deviation) from the pricing rule $(*)$ goes to zero as n becomes large. Thus, as the number of assets increases relative to the risk factors, expected returns will, on average, become closely approximated by the relation $a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z$.

Arbitrage Pricing Theory cont'd

- Also note that if there is a risk-free asset ($b_{iz} = 0, \forall\ z$), its return is approximately λ_0.

  Proof: For a given $n > k$, think of running a cross-sectional regression of the a_i's on the b_{iz}'s by projecting the dependent variable vector $a = [a_1\ \ a_2 \ldots a_n]'$ on the k explanatory variable vectors $b_z = [b_{1z}\ \ b_{2z} \ldots b_{nz}]'$, $z = 1, \ldots, k$. Define ν_i as the regression residual for observation i, $i = 1, \ldots, n$.
- Denote λ_0 as the regression intercept and λ_z, $z = 1, \ldots, k$, as the estimated coefficient on explanatory variable z.
- The regression estimates and residuals must then satisfy

$$a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + \nu_i \tag{30}$$

Regressing Assets' Expected Returns on Sensitivities

$$\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \lambda_0 + \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1k} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nk} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_k \end{bmatrix} + \begin{bmatrix} \nu_1 \\ \nu_2 \\ \cdot \\ \cdot \\ \cdot \\ \nu_n \end{bmatrix}$$

Arbitrage Pricing Theory cont'd

- By the properties of OLS, $\sum_{i=1}^{n} \nu_i = 0$ and $\sum_{i=1}^{n} b_{iz}\nu_i = 0$, $z = 1, ..., k$. Thus, we have shown that $(*)$, (i), and (ii) can be satisfied.
- The last and most important part of the proof is to show that (iii) must hold in the absence of asymptotic arbitrage.
- Construct a zero-net-investment arbitrage portfolio with the following investment amounts:

$$W_i = \frac{\nu_i}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \tag{31}$$

so that greater amounts are invested in assets having the greatest relative expected return deviation.

Arbitrage Pricing Theory cont'd

- The arbitrage portfolio return is given by

$$\tilde{R}_p = \sum_{i=1}^{n} W_i \widetilde{R}_i \tag{32}$$

$$= \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \widetilde{R}_i \right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \left(a_i + \sum_{z=1}^{k} b_{iz} \tilde{f}_z + \widetilde{\varepsilon}_i \right) \right]$$

- Since $\sum_{i=1}^{n} b_{iz} \nu_i = 0$, $z = 1, ..., k$, this equals

$$\tilde{R}_p = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \left(a_i + \widetilde{\varepsilon}_i \right) \right] \tag{33}$$

Arbitrage Pricing Theory cont'd

- Calculate this portfolio's mean and variance. Taking expectations:

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i a_i\right] \tag{34}$$

since $E[\widetilde{\varepsilon}_i] = 0$. Substitute $a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + \nu_i$:

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\lambda_0 \sum_{i=1}^{n} \nu_i + \sum_{z=1}^{k} \left(\lambda_z \sum_{i=1}^{n} \nu_i b_{iz}\right) + \sum_{i=1}^{n} \nu_i^2\right] \tag{35}$$

and since $\sum_{i=1}^{n} \nu_i = 0$ and $\sum_{i=1}^{n} \nu_i b_{iz} = 0$, this simplifies to

Arbitrage Pricing Theory cont'd

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \sum_{i=1}^{n} \nu_i^2 = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \nu_i^2} \tag{36}$$

- Calculate the variance. First subtract (34) from (33):

$$\tilde{R}_p - E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \widetilde{\varepsilon}_i\right] \tag{37}$$

- Since $E[\widetilde{\varepsilon}_i \widetilde{\varepsilon}_j] = 0$ for $i \neq j$ and $E[\widetilde{\varepsilon}_i^2] = s_i^2$:

$$E\left[\left(\tilde{R}_p - E\left[\tilde{R}_p\right]\right)^2\right] = \frac{\sum_{i=1}^{n} \nu_i^2 s_i^2}{n \sum_{i=1}^{n} \nu_i^2} < \frac{\sum_{i=1}^{n} \nu_i^2 S^2}{n \sum_{i=1}^{n} \nu_i^2} = \frac{S^2}{n} \tag{38}$$

Arbitrage Pricing Theory cont'd

- Thus, as $n \to \infty$, the variance of the portfolio's return goes to zero, and the portfolio's actual return converges to its expected return in (36):

$$\lim_{n \to \infty} \tilde{R}_p = E\left[\tilde{R}_p\right] = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \nu_i^2} \tag{39}$$

and absent asymptotic arbitrage opportunities, this certain return must equal zero. This is equivalent to requiring

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \nu_i^2 = 0 \tag{40}$$

which is condition (*iii*).

Arbitrage Pricing Theory cont'd

- We see that APT, given by the relation $a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z$, can be interpreted as a multi-beta generalization of CAPM.
- However, whereas CAPM says that its single beta should be the sensitivity of an asset's return to that of the market portfolio, APT gives no guidance as to what are the economy's multiple underlying risk factors.
- Empirical implementations include Chen, Roll and Ross (1986), Fama and French (1993), Heaton and Lucas (2000), Chen, Novy-Marx and Zhang (2011).
- We will later develop another multi-beta asset pricing model, the Intertemporal CAPM (Merton, 1973).

Summary

- The CAPM arises naturally from mean-variance analysis.
- CAPM and APT can also be derived from arbitrage arguments and linear models of returns.
- Arbitrage pricing arguments are very useful for pricing complex securities, e.g. derivatives.